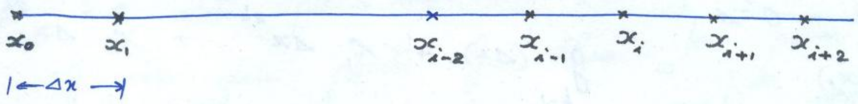
LECTURE - 31

NUMERICAL INTEGRATION . . . 15-10-2012

Last week you have seen the various Newton-Cotes Integration Formulas.

* Now recall the difference formulas:

Consider the data line as below:



You are aware that if the data interval for x is used as Δx, then the centered difference formula:

$$f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2)$$

i.e. $$f'(x_i) = g_1(\Delta x) + c\,\Delta x^2 + O(\Delta x^4)$$

Again if someone is taking the interval size as $2\Delta x$,

$$\therefore \quad f'(x_i) = \frac{f_{i+2} - f_{i-2}}{4\Delta x} + O(\cancel{2}\Delta x^2)$$

i.e. $$f'(x_i) = g_1(2\Delta x) + 4c\,\Delta x^2 + O(\Delta x^4)$$

We can write this as:

$$4f'(x_i) - f'(x_i) = 4g(\Delta x) - g(2\Delta x) \rightarrow O(\Delta x^4)$$

i.e. $$f'(x_i) = \frac{4g_1(\Delta x) - g_1(2\Delta x)}{3} + O(\Delta x^4)$$

i.e. We got an expression for first derivative using centered difference formula with $O(\Delta x^4)$.
(i.e. fourth-order approximations).
That is, we extrapolated the formula from

$g(\Delta x)$ and $g(2\Delta x)$ → two second order

expressions:

⇒ The same can be expressed in a general form for a first derivative from order of approximation $O(\Delta x^{2k})$
~~and O Δ~~

$$f'(x_i) = g_{k-1}(\Delta x) + c_1\,\Delta x^{2k} + c_2\,\Delta x^{2k+2} + \ldots$$

$$f'(x_i) = g_{k-1}(2\Delta x) + 4^k c_1\,\Delta x^{2k} + c_2\,4^{k+1}\,\Delta x^{2k+2} + \ldots$$

i.e. $$f'(x_i) = \frac{4^k\, g_{k-1}(\Delta x) - g_{k-1}(2\Delta x)}{4^k - 1} + O(\Delta x^{2k+2})$$

Gaussian Quadrature

You recall $I = \int_a^b f(x)\, dx$

→ If you have 'k' data points, then the maximum degree of polynomial that can be used to approximate the function $f(x)$ is $(k-1)$ degree.

→ This $(k-1)^{th}$ degree polynomial passes through all k data points with x_0 and x_n as the end points

→ If 'n' data points from these 'k' points are considered (inside), then definitely an $(n+1)^{st}$ degree polynomial should pass through these n-points.

→ This $(n+1)^{th}$ degree polynomial is fit to these n-points and integrated

$$I = \int_a^b f(x)\, dx = \sum_{i=1}^{n} C_i\, f(x_i)$$

where x_i locations at which the integrand function $f(x)$ is known, C_i → weighing factors.

→ On integrating, the degree raises by one.

∴ An additional degree of freedom exist.

→ So if 'n' points are used:

2n parameters are present $\quad x_i \rightarrow i = 1, 2, \ldots, n$
$\quad C_i \rightarrow i = 1, 2, \ldots, n$

There are $2n$ parameters.

With this known $2n$ parameters one can visualise a $(2n-1)^{th}$ degree polynomial

→ By choosing values of x_i we can obtain numerical integration methods.

Gaussian Quadrature is formed in such a way that the integral of the polynomial of $2n-1$ degree is an <u>exact value</u>.

Consider the function $F(t)$ and it is to be integrated between -1 and 1.

$$I = \int_{-1}^{1} F(t)\, dt = \sum_{i=1}^{n} C_i F(t_i)$$

First

Let us consider $n = 2$.

$\therefore$ The parameters are t_1, t_2, C_1 and C_2 (4 parameters)

$\therefore 2n-1 = 3$

I is exact for the following

$F(t) = 1$	An zero degree polynomial
$F(t) = t$	I^{st} degree
$F(t) = t^2$	II^{nd} degree
$F(t) = t^3$	III^{rd} degree

$$\therefore I\ (F(t) = 1) = \int_{-1}^{1} 1 . dt = \underline{\underline{2}}$$

$$I\,(F(t) = t) = \int_{-1}^{1} t \, dt = 0$$

$$I\,(F(t) = t^2) = \int_{-1}^{1} t^2 \, dt = \frac{2}{3}$$

$$I\,(F(t) = t^3) = \int_{-1}^{1} t^3 \, dt = 0$$

So you have exact integral values for these polynomials in the limit -1 to 1.

As $n = 2$, we are interested in finding those two points or values of 't', which can be utilised for integrating $\int_{-1}^{1} F(t)\, dt$.

So you have

$$\boxed{\begin{aligned} 2 &= C_1 \times 1.0 + C_2 \times 1.0 \\ 0 &= C_1 t_1 + C_2 t_2 \\ \frac{2}{3} &= C_1 t_1^2 + C_2 t_2^2 \\ 0 &= C_1 t_1^3 + C_2 t_2^3 \end{aligned}}$$

You need to solve these four equations to obtain

$C_1 = 1$

$C_2 = 1$

$t_1 = -\frac{1}{\sqrt{3}}$

$t_2 = \frac{1}{\sqrt{3}}$

$$\therefore I = \int_{-1}^{1} F(t)\, dt = \sum_{i=1}^{n} C_i F_i$$

$$= F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

That is the continuous integration $\int_{-1}^{1} F(t)\,dt$ is now approximated as $F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$ the sum of two function values.

⇒ As our problem of interest is to find

$$I = \int_a^b f(x)\,dx$$

→ Convert the limits suitably to the form

$$\int_{-1}^{1} F(t)\,dt$$

Let $x = mt + c$

At $x = a$, $t = -1$

$x = b$, $t = +1$

$\therefore\ m = \frac{b-a}{2}$, and $c = \frac{b+a}{2}$

$$\therefore\ x = \frac{b-a}{2} t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}\,dt$$

$$\therefore\ I = \int_a^b f(x)\,dx = \int_{-1}^{1} f(mt+c)\,dt\ \frac{b-a}{2}$$

$$\therefore\ I = \frac{b-a}{2} \int_{-1}^{1} f(mt+c)\,dt$$

$$= \frac{b-a}{2} \int_{-1}^{1} F(t)\,dt$$

$$I = \frac{b-a}{2} \left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \right]$$

Similarly based on the number of points within the given limits, if one selects, the integration schema varies.

→ As observed if one wants to use information from 2-point Gaussian quadrature

$$I = \frac{b-a}{2}\left[F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \right]$$

One can go for 3-point, 4-point Gaussian quadrature etc.

n No. of Points	t_i	C_i	Order
2	$-1/\sqrt{3}$	1	3
	$+1/\sqrt{3}$	1	
3	$-\sqrt{0.6}$	5/9	5
	0	8/9	
	$\sqrt{0.6}$	5/9	
4	−0.8611363116	0.3478548451	7
	−0.3399810436	0.6521451549	
	0.3399810436	0.6521451549	
	0.8611368116	0.3478548451	

<u>Example</u>

A function $f(x) = \frac{1}{x}$ is given to you. Evaluate $\int_{3.1}^{3.9} f(x)\,dx$.

$$\int_a^b f(x)\,dx = \int_{3.1}^{3.9} \frac{1}{x}\,dx$$

Let us use two point Gaussian quadrature:

$$I = \frac{3.9-3.1}{2}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$m = \frac{b-a}{2} = \frac{3.9-3.1}{2} = 0.4$$

$$c = \frac{b+a}{2} = 3.5$$

$$\therefore x = 0.4t + 3.5, \quad F(t) = \frac{1}{0.4t+3.5}$$

$$\therefore I = 0.4\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \underline{0.229571}$$

$\Rightarrow$ You can check with $\underline{n=3}$.

$$I = 0.4\left[\frac{5}{9}\times F(-\sqrt{0.6}) + \frac{8}{9}\times F(0) + \frac{5}{9}\times F(\sqrt{0.6})\right]$$

$$= \underline{0.229574}$$

Multiple Integrals

If you have $I = \int_c^d \int_a^b f(x,y)\,dx\,dy$

Then $I = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_c^d F(y)\,dy$

i.e. Double integral evaluated in two steps. $F(y) = \int_a^b f(x,y)\,dx$

→ Evaluate $F(y)$ at any selected values of y by num. integration.

→ Evaluate $I = \int_c^d F(y)\,dy$ by numerical integrations.