CE 601 NUMERICAL METHODS

LECTURE - 3
30-07-2012

Last day we have seen

i) How to formulate system of linear algebraic equations to an engineering problem.

ii) Possible types of solution
- → Unique
- → No solution
- → Infinite solutions
- → Trivial solution

iii) Basic approach to solve
- → Direct Elimination methods
- → Iterative methods

iv) Asked you to revise basics of matrix properties.

Today we will see some of the basic knowledge required before embarking on solution procedures for linear system.

Matrix Multiplication

$[A] \rightarrow n \times m$ order

$[B] \rightarrow m \times r$

Then $[AB] = \sum_{k=1}^{m} a_{i,k} b_{k,j}$; $i = 1, 2, 3, \ldots, n$; $j = 1, 2, 3, \ldots, r$

* Conformable matrices are associative in multiplication

That is [A], [B], [C] matrices are there,

then $A(BC) = (AB)C$

$$\left.\begin{array}{l}[A] \rightarrow n \times m \\ [B] \rightarrow m \times r \\ [C] \rightarrow r \times p\end{array}\right\}$$

Then $\underbrace{\underset{n \times m}{A}\ \underset{m \times p}{(BC)}}_{n \times p \text{ matrix}}$ & $\underbrace{\underset{n \times r}{(AB)}\ \underset{r \times p}{C}}_{n \times p \text{ matrix}}$

* Square matrices are conformable.

[A], [B]

Then $[AB] = [C]_{n \times n}$

$[BA] = D_{n \times n}$

* $A(B + C) = AB + AC$

Destribution property.

* Division of matrix is not defined.

* If you have [A]

Then $A A^{-1} = I$

IIly $A^{-1} A = I$

If two matrices [A], [B] such that

$[AB] = I$

Then $B = [A]^{-1}$

If $[A][B] = [C]$

Then $[A] = [B]^{-1}[C]$

Matrix factorisation

→ To represent a matrix as product of two other matrices

e.g. $[A] = [B][C]$

→ You can form infinite number of combination $[B]$ and $[C]$ for $[A]$.

Matrix Partitioning

→ To group the elements of a matrix into sub-matrices.

In the System of Linear Algebraic Equations

$[A]\{x\} = \{b\}$

$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad ; \; i = 1, 2, \ldots, n$

You can conduct row operations for solving such linear algebraic systems

Q: What is determinant?

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

e.g. In a 2 × 2 matrix

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

Q: How can you determine a 3×3 determinant ?

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Recall your mathematics courses:

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23})$$
$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

You can also determine for a 3×3 determinant

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

The triple products need to be identified.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13}$$
$$- a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

This is called <u>diagonal method</u>.

* Diagonal method is not easy for matrices greater than 3×3 size.
* You may have to go for <u>method of co-factors</u>.
* Check number of summations required in method of cofactors.

Direct Elimination Methods

Before discussing the direct elimination method it is better to go through the direct process of obtaining solutions of linear systems.

- → This is because it will help us in understanding how tedious it is through direct method
- → How advantage it is to use elimination methods.

Cramer's Rule

In a system $[A]\{x\} = \{b\}$

where $x_j = \dfrac{\det(A^j)}{\det(A)}$; $j = 1, 2, \dots, n$

where

$$[A^j] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix} \end{array}$$

e.g. For a 2×2 system $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad ; \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

* You require many calculations to determine the numerator and denominator determinants. For matrices above 3×3 order the process is quite difficult.

The Elimination Process

* Eliminate one unknown from first equation
* Two unknown from second equation, etc.
* Finally you get an expression for only one unknown

Then back-substitute to get all the unknowns.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Row Operations

To employ elimination process, you need to perform row operations.

(i) In the system any row (i.e. equation) can be multiplied by a constant. It is not going to change the solution. This is Scaling.

(ii) We can inter-change the order of rows as per our convenience. This is Pivoting.

(iii) We can also replace any row (i.e. equation) by a ~~l~~ weighted linear combination of that row with any other row. This is elimination.

* Please see that row operations will not change the solution of the system of equations.

Q: Why do we require row operations?

→ To prevent divisions by zero, to avoid round-off, and to implement systematic elimination.

Consider the following example from Hoffman "Numerical Methods...

$$\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}$$

How you used to solve such linear system

Augmented Matrix

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ -20 & 40 & -20 & 20 \\ -20 & -20 & 130 & 20 \end{array}\right]$$

~~$R_2 = R_1 + 4R_{21}$~~ $R_2 = R_1 + 4R2$

~~$R_3 = R_1 + 4R_{31}$~~ $R_3 = R_1 + 4R3$

$$\Rightarrow \left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 140 & -100 & 100 \\ 0 & -100 & 500 & 100 \end{array}\right] \quad R_3 = 5R_2 + 7R_3$$

$$\left[\begin{array}{ccc|c} 80 & -20 & -20 & 20 \\ 0 & 140 & -100 & 100 \\ 0 & 0 & 3000 & 1200 \end{array}\right]$$

$$3000 x_3 = 1200$$
$$x_3 = 0.40$$

Backsubstitution:

$$140 x_2 - 100 \times 0.4 = 100$$
$$\text{or} \quad x_2 = 1.0$$
$$80 x_1 - 20 - 8 = 20$$
$$\text{or} \quad x_1 = 0.60$$

⇒ In the entire process you had carried out various row operations unknowingly from your under-graduate experience.

Q: Why <u>Pivoting</u> will be required ?

→ Usually the matrix in a linear system is square.

→ The element on the major diagonal is called → Pivot (for any equation).

→ As we have seen the example, if it happens that $a_{11} = 0$, then the method would have failed.

→ So to avoid that elements in major diagonal have to be made non-zero by

↳ Interchanging rows (equations)

↳ Interchanging columns (variables)

→ This is Pivoting.

→ If you use both rows and columns → Full Pivoting

→ If you use interchanging of rows only → Partial Pivoting.

<u>Advantage of Pivoting</u>

→ Eliminate zero pivot elements

→ Reduce round-off errors.

e.g.: From Hoffman (2001)

$$\begin{bmatrix} 0 & 2 & 1 \\ 4 & 1 & -1 \\ -2 & 3 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -3 \\ 5 \end{Bmatrix}$$

$a_{11} = 0$; Therefore pivoting is required.

In first column, the largest element is in row 2.

∴ Interchange Row 1 and Row 2

$$\left[\begin{array}{ccc|c} 4 & 1 & -1 & -3 \\ 0 & 2 & 1 & 5 \\ -2 & 3 & -3 & 5 \end{array}\right] \quad R_3 = R_1 + 2R_3 \Rightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & -3 \\ 0 & 2 & 1 & 5 \\ 0 & 7 & -7 & 7 \end{array}\right]$$

$$R_3 = 7R_2 - 2R_3 \quad \Rightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & -3 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 21 & 21 \end{array}\right]$$

$x_3 = 1.0$

$2x_2 + 1.0 = 5$

or $x_2 = 2.0$

$4x_1 + 2 - 1 = -3$

$x_1 = -1$

Q: Why Scaling will be required?

→ Mainly to reduce round off errors.

→ Before elimination is done, for the first column the elements in the first column are scaled by the largest element in the corresponding row.

→ After scaling, then pivoting to be carried out.

e.g.: From HOFFMAN (2001) Page 38

You know computers cannot store numbers as fractions. It can only store them according to the number of digits suggested.

$$\begin{bmatrix} 3 & 2 & 105 \\ 2 & -3 & 103 \\ 1 & 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 104 \\ 98 \\ 3 \end{Bmatrix}$$

The actual solution here is $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}$

Let us consider only three significant digits.

$$\left[\begin{array}{ccc|c} 3 & 2 & 105 & 104 \\ 2 & -3 & 103 & 98 \\ 1 & 1 & 3 & 3 \end{array}\right]$$

Diagonal elements not zero. ∴ No Pivoting required.

We need to do elimination

$$R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1 = R_2 - 0.667\, R_1$$

$$R_3 - \left(\frac{a_{31}}{a_{11}}\right) R_1 = R_3 - 0.333\, R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 105 & 104 \\ 0 & -4.33 & \overset{33.0}{\cancel{32.965}} & 28.6 \\ 0 & 0.334 & \overset{-32.0}{\cancel{-31.965}} & -31.6 \end{array}\right]$$

$$R_3 - \left(\frac{a_{32}}{a_{22}}\right) R_2 = R_3 - (-0.077)\, R_2 \Rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 105 & 104 \\ 0 & -4.33 & \overset{33.0}{\cancel{32.965}} & 28.6 \\ 0 & 0 & \overset{-29.5}{\cancel{-29.427}} & -29.4 \end{array}\right]$$

$$x_3 = 0.997$$

$$-4.334\, x_2 + 32.965 = 28.632$$

$$x_2 = 0.924, \quad x_1 = -0.844$$