Lecture 23: Multivariate polynomial approximations, Cubic splines

(24-Spt-2012)

POLYNOMIAL APPROX---

LECTURE 23
24-SPT-2012

Till now we discussed on polynomial approximations to fit the data points usually for functions that are uni-variate

i.e. $y = f(x)$

y is varying only w.r.t x.

What happens if your function is varying w.r.t more than one variable.

Multi-variate Approximations

i.e. $y = f(x, t)$

or $z = f(x, y)$, etc.

That is, z depends on both x and y.

How can we approximate a function from the available data points now?

(For using in interpolation, differentiation, integration, etc.)

For multi-variate polynomial approximations too, you can have your polynomial that passes through

→ All points (Exact Fit)

→ Only through some or no points (Approximate Fit).

In the exact fit, there are:

* Successive uni-variate polynomial approximation
* Direct multi-variate polynomial approximation

Successive Uni-variate Poly. Approx..

→ Here if $z = f(x, y)$ and if you have data points

	x_1	x_2	x_3	x_4
y_1	z_{11}	z_{12}	z_{13}	z_{14}
y_2	z_{21}	z_{22}	z_{23}	z_{24}
y_3	z_{31}	z_{32}	z_{33}	z_{34}
y_4	z_{41}	z_{42}	z_{43}	z_{44}

→ What you do is

* First you fit approximate polynomial

$z_i = f(x, y_i)$ (That is for any particular row, you fit $z_i = f(x, y_i)$)

i.e. $z_i = a + bx + cx^2 + dx^3 + \dots$

* Using this approximate function interpolate

$z_i = f(x^*, y_i)$ the $x = x^*$ where we need interpolation.

~~Evaluate the~~ after finding coeffts $a, b, c, d, \dots$

Example:
Use successive univariate polyn. approximation to find the relation of enthalpy 'h' with pressure (P) and temperature (T). The following observational data is available

Pressure (kPa) \ Temp (°C)	427	538	649
7929	3211	3489	3755
8274	3205	3487	3753
8618	3199	3482	3751

Enthalpy is in kJ/kg.

Also evaluate enthalpy at $T = 596\,°C$ and $P = 8446$ kPa

Soln

Let us have a quadratic polynomial approximation

Also let us go row by row

$$\therefore \quad h(7929, T) = a + bT + cT^2$$

~~$h(8274, T) = a + b$~~

i.e. $3211 = a + b \times 427 + c \times (427)^2$

$$= a + 427b + 182329c$$

Also $3489 = a + 538b + 289444c$

$$3755 = a + 649b + 421201c$$

You can solve for a, b, and c

$$a = \cancel{2029.7}\ 2.029706 \times 10^3$$

$$b = 2.974433$$

$$c = -0.00048697$$

$$\therefore \quad h(7929, T) = 2.029706 \times 10^3 + 2.974433\,T - 0.00048697\,T^2$$

$$\therefore \quad h(7929, 596) = 3629.50 \quad kJ/kg$$

III) $h(8274, T) = 1.9710286 \times 10^3 + 3.167113\,T - 0.000649298\,T^2$

and $h(8618, T) = 1.979827 \times 10^3 + 3.097800\,T - 0.0005681357\,T^2$

Also $h(8274, 596) = 3628$ kJ/kg

and $h(8618, 596) = 3624$ kJ/kg

Now you have three data points $h(7929, 596)$, $h(8274, 596)$ and $h(8618, 596)$.

Quadratic Polynomial approximation to this will give

$$h(P, 596) = d + eP + gP^2$$

i.e.

$$3629.5 = d + 7929\,e + 62869041\,g$$
$$3628 = d + 8274\,e + 68459076\,g$$
$$3624 = d + 8618\,e + 74269924\,g$$

$d = 2.970786 \times 10^3$

$e = 0.16685$

$g = -0.00001056616$

$\therefore h(P, 596) = 2.970786 \times 10^3 + 0.16685\,P - 0.00001056616\,P^2$

$\therefore h(8446, 596) = \underline{3626.27}$ kJ/kg

There are also direct method for multi-variate approximation

Direct Multi-variate Polynomial Approx

In the same example $h\ (P, T)$

we can approximate ~~quadratic~~ high-order polynomial

$$h(P,T) = a + bP + cT + dPT + eP^2 + fT^2 + gP^2T + hPT^2 + iT^3 + jP^3 + \cdots$$

⇒ The number of data points should be atleast equal to number of coefficients

⇒ In the example if we use linear direct multi-variable polynomial

$$h = a + bT + cP + dTP$$

⇒ You can find the coefficients.

Cubic Splines

Based on the no. of data points, we can fit polynomial approximation.

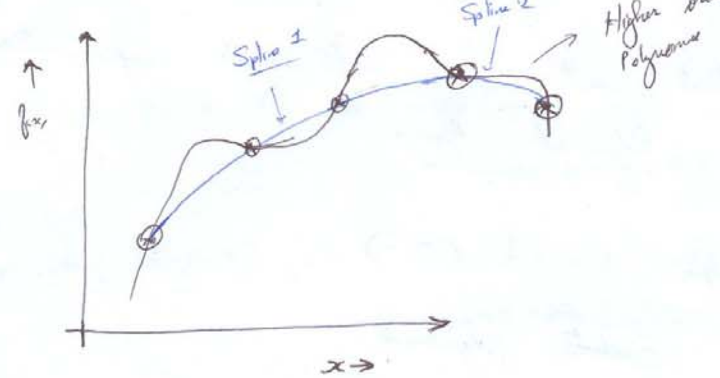
x	$f(x)$
x_0	f_0
x_1	f_1
x_2	f_2
$\vdots$	$\vdots$
x_n	f_n

→ Higher the order of polynomials, it passes through larger no. of points.

→ For e.g. the n^{th} degree polynomial will pass through all $(n+1)$ points.

→ However, the polynomial you created may be not accurate.

→ It may have troughs and crests between two data points. ~~That is, it may not be~~



→ If we can fit lower-degree polynomials between the sub-sets of data points and if they can be consistent it will give better result.

→ Such polynomials are called Splines.

→ Cubic ~~p~~ splines are third degree polynomials connecting data points.

There are $n+1$ points

n intervals

0 and n are boundary points

∴ There are $(n-1)$ interior grid points.

x_i $(i = 1, 2, 3, \ldots, n-1)$.

Cubic Spline to be fitted for each interval. ∴ At any interval ~~point~~

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

That is:

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

is the cubic spline at any interval $i = 1, 2, 3, \ldots, n$

Also $x_i \le x \le x_{i+1}$

→ Each cubic spline has four coefficients

→ There are n cubic splines

→ ∴ $4n$ coefficients

Now:

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

$$\therefore f_i'(x) = b_i + 2c_i x + 3d_i x^2$$

$$f_i''(x) = 2c_i + 6d_i x \quad \text{(Linear Equation).}$$

Using Lagrange polynomial approx.

$$f_i''(x) = \frac{x - x_i}{x_{i-1} - x_i} f_{i-1}'' + \frac{x - x_{i-1}}{x_i - x_{i-1}} f_i''$$

$$\therefore f_i'(x) = \frac{\frac{x^2}{2} - x x_i}{x_{i-1} - x_i} f_{i-1}'' + \frac{\frac{x^2}{2} - x x_{i-1}}{x_i - x_{i-1}} f_i'' + C$$

and

$$f_i(x) = \frac{\frac{x^3}{6} - \frac{x^2}{2} x_i}{x_{i-1} - x_i} f_{i-1}''(x) + \frac{\frac{x^3}{6} - \frac{x^2}{2} x_{i-1}}{x_i - x_{i-1}} f_i'' + Cx + D$$