Lecture 21: Divided Difference Polynomials, Newton's forward difference polynomial

(11-Spt-2012)

POLYNOMIAL APPROXIMATIONS (Contd...) →

LECTURE - 21
11 - SPT - 2012

In the last class, we discussed on:

→ Polynomial Approximations (in general)
→ Synthetic Division & Nested Algorithm
→ Direct - Fit Polynomials
→ Lagrange polynomials

* We saw that Direct - Fit Polynomials need to evaluate coefficients by solving system of equations. Can be used mostly in places where the discrete data are measured at equal intervals.

* To overcome such problems, Lagrange polynomials are used.

Today we will discuss on Divided - Difference tables & polynomials.

Divided Differences

→ Defined as ratio of the difference in the function values at two points divided by the difference in the values of the corresponding independent variable.

→ ~~Here~~ Let us go through the tabular data of x and $f(x)$

x_i	f_i
x_0	f_0
x_1	f_1
x_2	f_2
$\vdots$	$\vdots$
x_n	f_n

There are $(n+1)$ data points.
Divided difference can be evaluated at any data point.

e.g. The first Divided difference at any point i is

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

That is first divided difference at $i = 0$ is

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

$\Rightarrow$ The second divided difference at 'i' is defined as:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$\Rightarrow$ Third divided difference at point i

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}$$

You can see that

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{f_0 - f_1}{x_0 - x_1} = f[x_1, x_0]$$

→ For $i = 0$, in general we can state the n^{th} divided difference as:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots x_{n-1}]}{(x_n - x_0)}$$

⇒ Here onwards we will represent the order of divided difference as superscript

$$\therefore \quad f[x_0, x_1] = f_0^{(1)}$$

$$f[x_i, x_{i+1}] = f_i^{(1)}$$

Illy $f[x_i, x_{i+1}, x_{i+2}] = f_i^{(2)}$

and $f[x_i, x_{i+1}, \ldots, x_{i+n}] = f_i^{(n)}$ in general form.

x_i	f_i	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
x_0	f_0	$f_0^{(1)}$	$f_0^{(2)}$	
x_1	f_1	$f_1^{(1)}$	$f_1^{(2)}$	$f_0^{(3)}$
x_2	f_2	$f_2^{(1)}$		
x_3	f_3			

Therefore, one can tabulate the divided differences for any data point.

⇒ If you are given the same example. $d = f(t)$

t_i	d_i	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(2)}$	$f_i^{(4)}$	$f_i^{(5)}$
0.0	0.0					
		5.0				
10.0	50.0		0.25			
		10.0		0		
20.0	150.0		0.25		0	
		15.0		0		0
30.0	300.0		0.25		0	
		20.0		0		
40.0	500.0		0.25			
		25.0				
50.0	750.0					

Divided Difference Polynomial

(In interpolation, differentiation, integration, etc.)

⇒ The ~~divided~~ ~~diffe~~ polynomial $P_n(x)$ can be given as a power series with its coefficients as divided differences $f_i^{(n)}$. ∴ At any point i,

$$P_n(x) = f_i^{\bullet} + (x - x_0) f_i^{(1)} + (x - x_0)(x - x_1) f_i^{(2)} + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1}) f_i^{(n)}$$

This is a n^{th} degree polynomial passing through the $(n+1)$ data points.

It can be proved that $P_n(x)$ passes through these $(n+1)$ data points.

e.g: $P_n(x_0) = f_i + 0 \times f_i^{(1)} + 0 = f_i = f_0$

$P_n(x_1) = f_0 + (x_1 - x_0) f_0^{(1)} = f_0 + (x_1 - x_0) \dfrac{f_1 - f_0}{x_1 - x_0} = f_1$

etc.

$\Rightarrow$ In the working example, if we want to interpolate $f(t = 17\,s)$, then we can start at $i = 0$, $f_0 = 0.0$

$$P_n(17) = f_0 + (x - x_0) f_0^{(1)} + (x - x_0)(x - x_1) f_0^{(2)} + (x - x_0)(x - x_1)(x - x_2) f_0^{(3)} + (x - x_0) \ldots (x - x_3) f_0^{(4)} + (x - x_0)(x - x_1) \ldots (x - x_4) f_0^{(5)}$$

$$= 0.0 + (17 - 0) \times 5.0 + (17 - 0)(17 - 10) \times 0.25 + 0$$

$$= 114.75 \text{ m}$$

* Till now we discussed on divided difference and divided difference tables.

Difference Tables

* One can also form difference tables for the given data. Here we consider only difference in function values.

x_i	f_i		
x_0	f_0		
		$(f_1 - f_0)$	
x_1	f_1		$(f_2 - 2f_1 + f_0)$
		$(f_2 - f_1)$	
x_2	f_2		$f_3 - 2f_2 + f_1$
		$(f_3 - f_2)$	
x_3	f_3		

One can define forward difference operator : Δ

Backward diff. operator ∇

Centered diff operator δ

$$\Delta y_i = y_{i+1} - y_i$$

$$\nabla y_i = y_i - y_{i-1}$$

$$\nabla y_{i+1} = y_{i+1} - y_i$$

$$\delta y_{i+1/2} = y_{i+1} - y_i$$

x	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		
x_3	y_3			

Newton Forward - Difference Polynomial

If there are $(n+1)$ data points

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

Then the n^{th} degree polynomial is unique.
This polynomial that passes through the $(n+1)$ data points is given by:

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots + \frac{s(s-1)(s-2)\cdots(s-(n-1))}{n!}\Delta^n f_0$$

where $s = \dfrac{(x - x_0)}{\Delta x}$

also called interpolating variable.

$\therefore \quad x = x_0 + s\Delta x$

As $\quad s = \dfrac{x - x_0}{\Delta x} \quad \rightarrow$ linear in x.

$P_n(x) \rightarrow$ is also an n^{th} degree polynomial.

<u>When $s = 0$</u>;

$x = x_0$, we have

$P_n(x_0) = f_0$.

<u>When $s = 1$</u>;

$s = \dfrac{x - x_0}{\Delta x} = 1$, $\quad \therefore \quad x = x_0 + \Delta x = x_1$

$$P_n(x_1) = f_0 + \Delta f_0$$
$$= f_0 + (f_1 - f_0) = f_1$$

Similarly $P_n(x)$ passes through all $(n+1)$ data points.

This is Newton's forward difference polynomial.

Using binomial coefficient

$$\binom{s}{i} = \frac{s(s-1)(s-2)\cdots(s-(i-1))}{i!}$$

$$P_n(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \cdots$$