Lecture 18: Solution of Non-Linear Equations

Newton's method for finding single root and multiple roots of polynomials (04-Spt-2012)

SOLUTIONS OF NON-LINEAR EQUATIONS (CONTD...)

LECTURE-18
04-SPT-2012

In the last class we discussed on

* Secant Method
* Difference between Secant method and Regula-Falsi method
* Muller's method
* Introduced solutions of polynomials.
  → You can use Newton's methods

Polynomial Solutions

As discussed yesterday, polynomials are non-linear equations (All non-linear equations are not polynomials).
They have the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $P_n(x) \rightarrow n^{th}$ degree polynomial

The open domain methods like

→ Newton's method
→ Secant method
→ Muller's method, etc. can be used to find the roots of the polynomials.

We have seen that the order of convergence of Newton's method is quadratic.

Recall,
$$e_{i+1} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} e_i^2$$

For the polynomial $f(x) = x^3 - 3x^2 + 4x - 2 = P_3(x)$, if we want to find a simple root, we can use Newton's method as follows:

$$f(x) = x^3 - 3x^2 + 4x - 2$$
$$f'(x) = P_3'(x) = 3x^2 - 6x + 4$$

Newton's method $\quad x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Let us start at $i = 0$, $x_0 = 2.00000$

We are giving convergence criteria $|f(x_i)| \leq 1 \times 10^{-4}$

For $i = 1$,

$f(x_0) = 2$, $f'(x_0) = 4$

$\therefore x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2.00000 - \dfrac{2}{4} = 1.50000$

For $i = 2$

$f(x_1) = 0.625$, $f'(x_1) = 1.75$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.50000 - \dfrac{0.625}{1.75} = 1.14286$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	2.00000	2	4	1.50000
1	1.50000	0.625	1.75000	1.14286
2	1.14286	0.14578	1.06123	1.00549
3	1.00549	0.00549	1.00009	1.00000
4	1.00000	0.00000	–	–

The solution here is $\alpha = 1.00000$
for $f(x) = x^3 - 3x^2 + 4x - 2$

As you know $f(x) = x^3 - 3x^2 + 4x - 2$ is a third degree polynomial. $\therefore$ Therefore there exists three roots α_1, α_2, and α_3.

One root is $\alpha_1 = 1.00000$

→ To find other roots we can use polynomial deflation

i.e. if a n^{th} degree polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is having one root $x = \alpha$.

Then we can express

$$P_n(x) = (x - \alpha)\, Q_{n-1}(x)$$

where the polynomial

$$Q_{n-1}(x) = b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}$$

and

$$b_n = a_n$$

$$b_i = a_i + \alpha b_{i+1}$$

$$i = n-1, n-2, \ldots, 1$$

$\therefore$ In the given problem;

$$P_3(x) = x^3 - 3x^2 + 4x - 2 \quad = -2 + 4x - 3x^2 + x^3$$

$\alpha = 1.00000$ is a simple root of $P_3(x)$

$$\therefore P_3(x) = (x - 1.00000)\, Q_2(x)$$

where $Q_2(x) = b_1 + b_2 x + b_3 x^2$

$$b_3 = a_3 = 1.0000$$

$$b_2 = a_2 + \alpha b_3 = -3 + (1 \times 1.000) = -2.000$$

$$b_1 = a_1 + \alpha b_2 = 4 + (1 \times -2) = 2.0000$$

$$\therefore \quad Q_2(x) = b_1 + b_2 x + b_3 x^2$$
$$= 2 - 2x + x^2$$

$Q_2(x) \rightarrow$ is a second degree polynomial for which roots can be found as:

$$\alpha_2', \alpha_3 = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}}{2b_3} = \frac{2 \pm \sqrt{4 - 4 \times 2 \times 1}}{2}$$
$$= \frac{1}{2}(2 \pm \sqrt{-4}) = 1 \pm i$$

$\alpha_2 = 1 - i$
$\alpha_3 = 1 + i$

Q: What happens to Newton's method, if you have multiple roots?
i.e. For a given polynomial $P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$
a certain root α may be repeating 'm' times.

: $\rightarrow$ For such instances, the quadratic convergence rate in Newton's method will reduce to linear convergence rate.

For the following example:

$$x^3 - 1.2502\,x^2 - 1.5625\,x + 1.95344 = 0$$

If you try to find a root, 'α'

Start $x_0 = 1.50000$, $\therefore f'(x) = 3x^2 - 2.5004\,x - 1.5625$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	1.50000	0.17174	1.4369	1.38048
1	1.38048	0.04472	0.70292	1.31686
2	1.31686	0.01144	0.34718	1.28391
3	1.28391	0.00290	0.17249	1.26710
4	1.26710	0.00073	0.08587	1.25860
5	1.25860			
⋮				

You can see while evaluating $f(x_i)$, it is not quadratically converging to zero after some iterations. It is almost linearly convergent.

* This is because the solution (or root) that is approaching may be multiple in nature.

∴ In a general form for a polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = f(x)$$

if α is a solution that repeats m times, this can be said as below

$$\lim_{x \to \alpha} \frac{|f(x)|}{|x-\alpha|^m} = c \quad \text{where } c \neq 0$$

Now $f(x) = P_n(x) = (x-\alpha)^m A(x)$

where $A(x)$ is such that $A(\alpha) \neq 0$

i.e. $x = \alpha$ is not a root of $A(x)$.

The Newton's method become:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{A(x_i)(x_i-\alpha)^m}{\left[m A(x_i)(x_i-\alpha)^{m-1} + (x_i-\alpha)^m A'(x_i)\right]}$$

i.e.

$$x_{i+1} - \alpha = x_i - \alpha - \frac{A(x_i)(x_i-\alpha)^m}{\left(m A(x_i)(x_i-\alpha)^{m-1} + A'(x_i)(x_i-\alpha)^m\right)}$$

$$= (x_i - \alpha)\left(1 - \frac{1}{m + \frac{(x_i-\alpha) A'(x_i)}{A(x_i)}}\right)$$

i.e. $$(x_{i+1} - \alpha) = (x_i - \alpha)\left(1 - \frac{1}{m + \frac{(x_i - \alpha) A'(x_i)}{A(x_i)}}\right)$$

When i is very large no.

$$e_{i+1} \approx e_i \left(1 - \frac{1}{m}\right)$$

Hence the convergence rate is linear.

→ To restore the second-order convergence, we can write

Say if $x = \alpha$ occurs m times, then

we can represent a transformed function $(f(x))^{1/m}$

such that this transformed function has the root $x = \alpha$.

$$\therefore \quad x_{i+1} = x_i - \frac{(f(x_i))^{1/m}}{\frac{d}{dx}(f(x_i))^{1/m}}$$

$$= x_i - \frac{(f(x_i))^{1/m}}{\frac{1}{m}(f(x_i))^{\frac{1}{m}-1} f'(x_i)}$$

$$\boxed{x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}}$$

So you should know the multiplicity m for α.

Example:

In the previous example, we saw there may exist double root.
Let us say $m = 2$

$$x^3 - 1.2502\,x^2 - 1.5625\,x + 1.95344 = 0$$

Sol. Assuming $x_0 = 1.50000$

$$f(x) = x^3 - 1.2502\,x^2 - 1.5625\,x + 1.95344$$

$$f'(x) = 3x^2 - 2.5004\,x - 1.5625$$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	1.5000	0.17174	1.4369	1.26096
1	1.26096	0.00030	1.	

You can see convergence is faster.

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* Another process if you don't know multiplicity is

Assuming $u(x) = \dfrac{f(x)}{f'(x)}$

$$\therefore\ f(x) = (x-\alpha)^m A(x)$$

where $A(\alpha) \neq 0$.

Now
$$u(x) = \frac{(x-\alpha)^m A(x)}{m(x-\alpha)^{m-1} A(x) + (x-\alpha)^m A'(x)}$$

$$u(x) = \frac{(x-\alpha)\,A(x)}{m\,A(x) + (x-\alpha)\,A'(x)}$$

It is quite evident $u(x)$ have the solution $x = \alpha$.

$$\therefore \quad x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$u'(x) = \frac{f(x_i)\,f'(x_i)}{(f'(x_i))^2 - f(x_i)\,f''(x_i)} \qquad \frac{f'(x_i)\,f'(x_i) - f(x_i)\,f''(x_i)}{(f'(x_i))^2}$$

$\therefore$ Now Newton's method becomes.

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)\,f'(x_i)}{(f'(x_i))^2 - f(x_i)\,f''(x_i)}}$$

You can check in the same example problem.

Example

$$f(x) = x^3 - 1.2502\,x^2 - 1.5625\,x + 1.95344$$

$$f'(x) = 3x^2 - 2.5004\,x - 1.5625$$

$$f''(x) = 6x - 2.5004$$

Assume $x_0 = 1.50000$

i	x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$	x_{i+1}
0	1.50000	0.17174	1.4369	6.4996	1.23981
1	1.23981				