Lecture 17: Solution of Non-Linear Equations

Secant method, Muller's method, Polynomials as non-linear equations (03-Spt-2012)

EE 601

LECTURE - 17
03-SPT- 2012

SOLUTIONS OF NON-LINEAR EQUATIONS (Contd...)

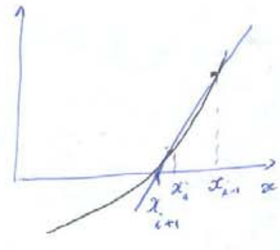
In the last class, we discussed on
* Newton's method (or Newton-Raphson method)
where you can improve your solution to non-linear equation by iterative technique.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

* We also saw that Newton's method is quadratic convergent (i.e order of convergence is of degree 2). That means, it converges faster compared to fixed-point iteration.

* If there is difficulty in evaluating $f'(x_i)$ for any function, then you may see that Newton's method is not feasible.

* In that case we suggested that secant method is more useful (or appropriate).



Secant method approximates function $f(x)$ by a straight line $g(x)$ between two points x_{i-1} & x_i.

∴ We get

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{g'(x_i)}}$$

You can elaborate this method:

In the straight line $g(x)$, the slope is

$$g'(x_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Also $g'(x_{i+1}) = \dfrac{g(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$

$$= \frac{-f(x_i)}{x_{i+1} - x_i}$$

∴ Equating we get:

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{-f(x_i)}{x_{i+1} - x_i}$$

or Improved value:

$$x_{i+1} - x_i = \frac{-f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

i.e. $$\boxed{x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

See the similarity in the mathematical form of Secant method and Regula Falsi method:

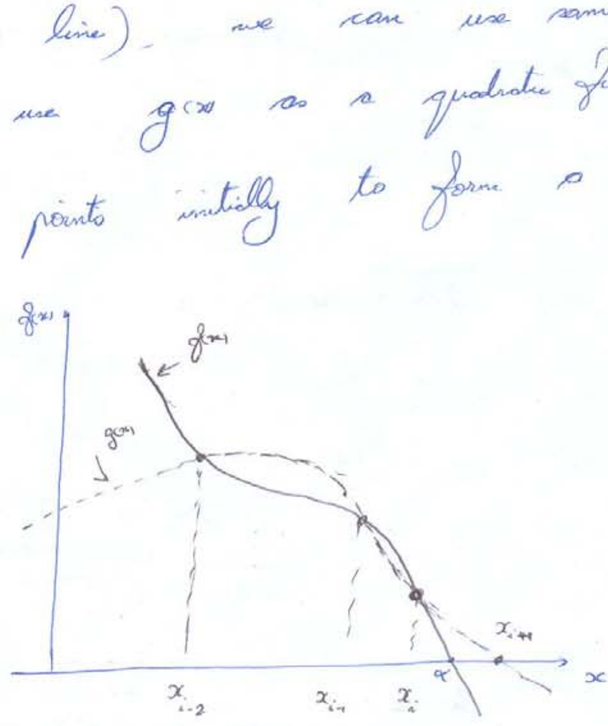
→ But they are entirely two different form.
Also one is closed domain method (Regula - Falsi) where the improved value x_{i+1} is between the domain $[x_{i-1}, x_i]$;
The other is open domain (Secant method), where x_{i+1} is not necessarily be in between x_{i-1} and x_i.

Muller's Method

As seen in Secant method, where we approximated $f(x) \approx g(x)$ (a straight line), we can use same principle, but instead use $g(x)$ as a quadratic function.

* You require minimum three points initially to form a quadratic relation

So there will be three known values x_{i-1}, x_{i-2}, and x_i at any iteration $(i+1)$.



In Newton's method

$$x_{i+1} = x_i - \frac{f(x)}{f'(x)}$$

$$f(x) \approx g(x) = a(x - x_i)^2 + b(x - x_i) + c$$

We have known points $(x_i, f(x_i))$, $(x_{i-1}, f(x_{i-1}))$, and $(x_{i-2}, f(x_{i-2}))$ ... At these points the quadratic function $g(x)$ is

$$\left.\begin{aligned} g(x_i) &= a \times 0 + b \times 0 + c = f(x_i) = c \\ g(x_{i-1}) &= a \times (x_{i-1} - x_i)^2 + b(x_{i-1} - x_i) + c = f(x_{i-1}) \\ g(x_{i-2}) &= a(x_{i-2} - x_i)^2 + b(x_{i-2} - x_i) + c = f(x_{i-2}) \end{aligned}\right\}$$

From these three relationships

$c = f(x_i)$

~~$b = \left(f(x_{i-2}) - f(x_i)\right)$~~

Assign
$$\Delta x_1 = x_{i-1} - x_i$$
$$\Delta x_2 = x_{i-2} - x_i$$
$$\delta f_1 = f(x_{i-1}) - f(x_i)$$
$$\delta f_2 = f(x_{i-2}) \; f(x_i)$$

i.e. $f(x_{i-1}) - f(x_i) = a(x_{i-1} - x_i)^2 + b(x_{i-1} - x_i)$

$$\delta f_1 = a\,\Delta x_1^2 + b\,\Delta x_1$$

ly $\qquad \delta f_2 = a\,\Delta x_2^2 + b\,\Delta x_2$

$$\therefore \quad a = \frac{\delta f_1\,\Delta x_2 - \delta f_2\,\Delta x_1}{\Delta x_1\,\Delta x_2\,(\Delta x_1 - \Delta x_2)}$$

$$b = \frac{\delta f_2\,\Delta x_1^2 - \delta f_1\,\Delta x_2^2}{\Delta x_1\,\Delta x_2\,(\Delta x_1 - \Delta x_2)}$$

i.e. $g(x) = a(x - x_i)^2 + b(x - x_i) + c$

Now at x_{i+1}, $g(x_{i+1}) = 0 = a(x_{i+1} - x_i)^2 + b(x_{i+1} - x_i)$

or. $(x_{i+1} - x_i)\left[a(x_{i+1} - x_i) + b\right] = -c$

or $a(x_{i+1} - x_i)^2 + b(x_{i+1} - x_i) + c = 0$

i.e. $$(x_{i+1} - x_i) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now considering the RHS term

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) \times \frac{\left(-b \mp \sqrt{b^2 - 4ac}\right)}{\left(-b \mp \sqrt{b^2 - 4ac}\right)}$$

$$= \frac{b^2 - (b^2 - 4ac)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)} = \frac{2c}{-\left(b \pm \sqrt{b^2 - 4ac}\right)}$$

or $$x_{i+1} = x_i - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

Convergence $|x_{i+1} - x_i| \leq \varepsilon_1$

or $|f(x_{i+1})| \leq \varepsilon_2$

Example. (Taken from Gupta & Srivastava, 2010)

Solve the non-linear function $f(x) = x^3 - 1.25x^2 - 1.562525x + 1.953 = 0$

Let the convergence criterion be 1×10^{-4}.

Soln. Given $f(x) = x^3 - 1.25x^2 - 1.56253x + 1.9531 = 0$

Let us initially assume three values of 'x'

Say x_0, x_{0-1}, and x_{0-2}.

i.e. At $x_{0-2} = 0$. we have $f = 1.9531$

Let us take $x_{0-1} = 0.25$, $f(0.25) = 1.49997$

At $x_0 = 0.50$, $f(0.5) = 0.98434$

Iteration	x_{i-2}	$f(x_{i-2})$	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	x_{i+1}
.	0	1.9531	0.25	1.49997	0.50	0.98434	

Iteration (1):

$c = f(x_2) = 0.98434$

$$b = \frac{\delta f_2\, \Delta x_1^2 - \delta f_1\, \Delta x_2^2}{\Delta x_1 \Delta x_2 (\Delta x_1 - \Delta x_2)} = \frac{0.96876\,(0.25)^2 - 0.51563\,(0.50)^2}{(0.50)(0.25)^2\,(0.25 - 0.50)}$$

$$=$$

$$a = \frac{\delta f_1\, \Delta x_2 - \delta f_2\, \Delta x_1}{\Delta x_1 \Delta x_2 (\Delta x_1 - \Delta x_2)} = \frac{0.51563 \times 0.50 - 0.96876 \times 0.25}{(0.25)(0.50)\,(0.25 - 0.50)} =$$

(Please continue the problem)

~~Using Polynomials to obtain solution to Non Linear Equation~~

Solutions of Polynomials

~~One of the~~ A type of non-linear equation is a polynomial.

n^{th} degree polynomial:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

→ From algebra, you know that there are n-roots for $P_n(x)$.

→ Roots can be real, or complex.

→ Roots can be single or multiple

First-degree Polynomial

$$P_1(x) = ax + b$$

If $x = \alpha$ is the exact root

then $\alpha = -\frac{b}{a}$.

Second-degree

$$P_2(x) = ax^2 + bx + c \neq 0$$

α_1, α_2 roots (say).

the $\alpha_1, \alpha_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Higher-degree Polynomials

It is difficult to find ~~solutions~~ roots by the method for higher degree polynomials.

A useful method is Polynomial Deflation.

→ We can use the Open Domain methods

* Newton's
* Secant
* Muller's etc.

To find roots of polynomials (non-linear).

Again revisit Newton's method for Polynomials

If $x = \alpha$ is the exact solution of $f(x) = $

Say $f(x) = P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

Then $f'(x) = P_n'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Example

i.e. $f(x) = P_3(x) = x^3 - 3x^2 + 4x - 2 = 0$

$\therefore f'(x) = P_3'(x) = 3x^2 - 6x + 4$

Let us start with initial guess $x = 2$

$\therefore x_0 = 2$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	$f(x_{i+1})$
0	2	2	4	1.5	
1	1.5	0.625	1.75	1.14286	
2	1.14286	0.14578	1.06123	~~0.13737~~ 1.00549	
3	1.00549	0.00549			

Continue

QUIZ 03-SPT-2012

Answer Only in Two (or Three Sentence Max).

1) What is the order of convergence of fixed-point iteration scheme?

2) Can you use Newton's method for the function $f(x) = 15\cos^2 x + 10\cos x$ using an initial estimate $x_0 = 0.000$. Why?

3) Are Secant Method & Regula Falsi method same? Why?

4) How many initial guess are required in Muller's method?