## Lecture 16: Solution of Non-Linear Equation

**Newton's method, Order of Convergence** **(30-Aug-2012)**

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LECTURE - 16

30-AUG-2012

Convergence Criteria in Fixed-Point Iteration

As discussed in last lecture, the fixed-point iteration suggest

$$x_{i+1} = g(x_i)$$

If $\alpha$ is the exact root of the function $f(x)$,

Then error in any iteration

$$e_{i+1} = x_{i+1} - \alpha = g(x_i) - g(\alpha)$$

Refer your Taylor's series for any function $p(x)$ w.r.t a point $x_0$.

$$p(x) = p(x_0) + \frac{1}{1!} p'(x_0)(x - x_0) + \frac{1}{2!} p''(x_0)(x - x_0)^2 + \cdots$$

The same logic is applied to describe $g(\alpha)$ w.r.t $x_i$

i.e.

$$g(\alpha) = g(x_i) + g'(x_i)(\alpha - x_i) + \frac{1}{2!} g''(x_i)(\alpha - x_i)^2 + \cdots$$

If you are truncating this series after the first order differential term, then you can write

$$g(\alpha) = g(x_i) + g'(\xi)(\alpha - x_i), \quad \text{where}$$

$$x_i \leq \xi \leq \alpha$$

or. Substituting this in the expression for $e_{i+1}$

$$e_{i+1} = g(x_i) - g(x_i) - g'(\xi)(\alpha - x_i) - \cdots$$

i.e. $e_{i+1} = g'(\xi)\, e_i$

Now for this process to be convergent, you need the error to decrease from previous iteration.

i.e. $$\left| \frac{e_{i+1}}{e_i} \right| = g'(\xi) < 1$$

(If this is $> 1$, the iteration diverges).

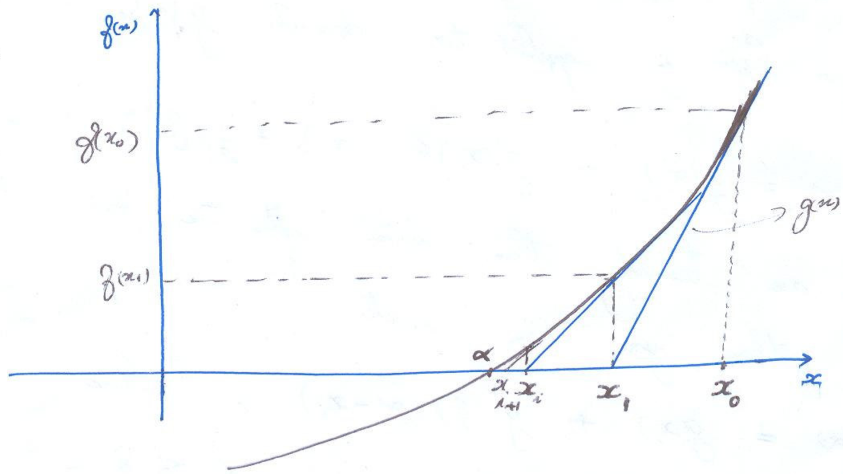
## Newton's Method

→ This method is also called Newton-Raphson method.

→ It is an iterative procedure to find a root of the non-linear function $f(x) = 0$.

→ In that start with initial guess of $x$ (say $x_0$)

→




→ Find the corresponding function value $f(x_0)$.

→ Draw a tangent line passing through the ~~graph~~ graphical point $(x_0, f(x_0))$.

→ This line intersects the $x$-axis at $x_1$ an improved value compared to $x_0$.

→ Again find $f(x_1)$ and draw tangent along $(x_1, f(x_1))$.

→ The improved point $x_2$ is got.

→ The process is repeated till convergence

→ At any general $(i+1)^{th}$ iteration → $x_{i+1}$ is obtained.

→ At convergence $f(x_{i+1}) \rightarrow f(\alpha)$

For the slope of the tangented straight line at any $(i+1)^{th}$ iteration

i.e. $$g'(x_i) = \frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

Note: → The slope of the straight line is also slope of the curve $f(x)$ at the point $(x_i, f(x_i))$.

$$\therefore \quad g'(x_i) = f'(x_i)$$

$$\therefore \quad f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

or $$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Repeat the process till convergence. $|x_{i+1} - x_i| \leq \varepsilon_1$

or $|f(x_{i+1})| \leq \varepsilon_2$, etc.

Example

Solve the function $f(x) = x^2 - 10x + 23$ using Newton-Raphson method.

Soln. The function is non-linear

$$f(x) = x^2 - 10x + 23 = 0$$

To identify a root of $f(x) = 0$.

Start with $x_0 = 4$

$$f(x) = x^2 - 10x + 23$$

$$\therefore f'(x) = 2x - 10$$

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $x_{i+1}$ |
|---|---|---|---|---|
| 0 | 4 | $-1$ | $-2$ | 3.50000 |
| 1 | 3.50000 | 0.25000 | $-3$ | 3.58333 |
| 2 | 3.58333 | 0.00695 | $-2.83334$ | 3.58578 |
| 3. | 3.58578 | 0.00002 | | |

The results conv. i.e. $|f(x_i)| \le 1 \times 10^{-4}$.

Regarding Error Criteria in N-R Method

Here again let us define

$$e_{i+1} = x_{i+1} - \alpha$$

Now $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

i.e. $x_{i+1} = g(x_i)$ ; where $g(x_n) = x_i - \dfrac{f(x_n)}{f'(x_n)}$

For convergence you require $|g'(\xi)| < 1$ $\quad x_i < \xi < \alpha$

(As discussed in fixed point iteration).

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\therefore g'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{[f'(x)]^2}$$

As obvious at exact convergence $x = \alpha$, $g(x) = g(\alpha)$

$$\therefore \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\text{i.e.} \quad x_{i+1} - \alpha = x_i - \alpha - \frac{f(x_i)}{f'(x_i)}$$

$$\text{i.e.} \quad e_{i+1} = e_i - \frac{f(x_i)}{f'(x_i)}$$

Expanding $f(\alpha)$ as Taylor's series w.r.t. $x_i$ and truncating after second-order term:

$$f(\alpha) = f(x_i) + f'(x_i)(\alpha - x_i) + \frac{1}{2} f''(\xi)(\alpha - x_i)^2 = 0$$

$$x_i \le \xi \le \alpha$$

$$\text{i.e.} \quad f(x_i) = e_i f'(x_i) - \frac{1}{2} f''(\xi) e_i^2$$

$$\text{or} \quad e_{i+1} = e_i - \frac{e_i \left( f'(x_i) - \frac{1}{2} f''(\xi) e_i \right)}{f'(x_i)}$$

$$= \frac{1}{2} \frac{f''(\xi)}{f'(x_i)} e_i^2$$

In the limit $i \to \infty$, $x_i \to \alpha$, $f'(x_i) \to f'(\alpha)$, $f''(\xi) \to f''(\alpha)$

$$\text{Then} \quad e_{i+1} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} e_i^2$$

Convergence is second order (or quadratic).

## Secant Method

Derivative function $f'(x)$ is ~~…~~ not possible to evaluate alternate to Newton's method.

That is Secant method is more appropriate in situations where you encounter $f'(x)=0$.

Therefore, in $x_{i+1}=x_i-\dfrac{f(x)}{f'(x)}$, the slope $f'(x)$ is substituted by the slope $g'(x)$ of the straight line $g(x)$, which is a secant line to the curve $f(x)$.