Lecture 14: Solution of Non-Linear Equation

Regula Falsi method, Fixed-point iteration **(28-Aug-2012)**

LECTURE 15
28 - AUGUST - 2012

SOLUTIONS OF NON-LINEAR EQUATIONS

As discussed yesterday for solving non-linear equations $f(x) = 0$, there are

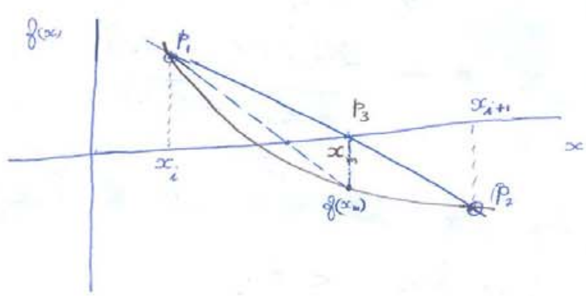
(i) Closed domain methods
(ii) Open domain methods

In closed domain method, we discussed on - Bisection (or Half-Interval) method.

* As you see, the method will converge to the result if the domain is bracketed appropriately initially.
* However, the method is very slow, as everytime it brackets the domain at the mid-point of the previous interval.
* To increase the speed of convergence, there is another method:

False Position (or Regula Falsi Method)

→ This method determines the domain at each iteration by not halving the interval.



→ let us the plot is 'x' vs. $f(x)$ and we need to find the solution, i.e. 'x' at which $f(x) = 0$.

→ Now identify two points on the curve $f(x)$ say P_1 and P_2.

where $P_1\ (x_i, f(x_i))$ and $P_2\ (x_{i+1}, f(x_{i+1}))$

and also $f(x_i)\, f(x_{i+1}) < 0$

i.e. Domain is bracketed within $[x_i, x_{i+1}]$.

→ Draw a straight line from P_1 to P_2 and this straight line cross x-axis at the point $P_3(x_m, 0)$.

→ This point $P_3\ (x_m, 0)$ is taken for the domain in the new iteration (rather than mid-point of x_i and x_{i+1}).

→ Slope of the straight line between P_1 and P_2

$$m = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

The slope can be evaluated between P_1 and P_3 as well

$$m = \frac{0 - f(x_i)}{x_m - x_i}$$

$$\therefore \quad \frac{-f(x_i)}{x_m - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\text{or} \quad x_m = x_i - \frac{f(x_i)\,(x_{i+1} - x_i)}{f(x_{i+1}) - f(x_i)}$$

→ Please note that here x_i and x_m will bracket the solution here i.e. $f(x_i)\, f(x_m) < 0$.

$\therefore$ The new domain will be $[x_i, x_m]$

→ If not the $f(x_m)\, f(x_{i+1}) < 0$.

Apply the same procedure to find the ~~slope~~ straight line, intersection between x_i and x_m.
Continue till x_m converges to the solution
i.e. $f(x_m) \approx 0.0$.

Example
Solve the non-linear equation $f(x) = x^2 - 10x + 23$

Soln. First we need to find the closed domain for the solution (or the domain in which solution is included).
For that find ~~two points~~ ~~x_i, x_{i+1}~~ value of x_i and x_{i+1} such that $f(x_i)\, f(x_{i+1}) < 0$

let us assume $x_i = 5$; $f(x_i) = -2$
$x_{i+1} = 7$; $f(x_{i+1}) = 2$

$\therefore$ Our domain is $[x_i, x_{i+1}] = [5, 7]$. and $x_m = x_i - \dfrac{f(x_i)(x_{i+1} - x_i)}{f(x_{i+1}) - f(x_i)}$

x_i	x_{i+1}	$f(x_i)$	$f(x_{i+1})$	x_m	$f(x_m)$
5	7	-2	2	6	-1
6	7	-1	2	6.33333	-0.22223
6.33333	7	-0.22223	2	6.40000	-0.04
6.4000	7	-0.04	2	6.41176	-0.00693
6.41176	7	-0.00693	2	6.41379	-0.00120
6.41379	7	-0.00120	2	6.41414	-0.00021

Say let us keep converge factor $|f(x)| \leq 1 \times 10^{-3}$. Then it is converged.
$x = 6.41414$.

→ The closed domain (bracketing) methods guarantee convergence.

→ However they are slower in convergence.

→ Due to this open domain methods are preferred.

OPEN DOMAIN METHODS

→ The roots are not bracketed as like in closed domain method.

→ Some of the methods are:

- Fixed point iteration
- Newton's method
- Secant method
- Muller's method

Fixed point iteration

As suggested earlier we need to solve the non-linear equation $f(x) = 0$

The $f(x) = 0$ equation can be possible to be rearranged as:

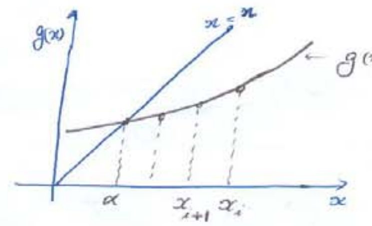
$$x = g(x)$$

eg: $f(x) = 2 - x^2$ can be given as $x^2 = 2$ or $x = \frac{2}{x}$

etc..

Using this philosophy, the fixed point iteration scheme works.

i.e. $x = g(x_0)$



* Initial guess x_i
* Evaluate $g(x_i)$
* Subsequently evaluate $x_{i+1} = g(x_i)$
* Procedure repeated till $|x_{i+1} - x_i| \leq \varepsilon_1$ (tolerance).
or $|f(x_{i+1})| \leq \varepsilon_2$ (tolerance)

Example.

Solve the function $f(x) = x^2 - 10x + 23$

Soln. $f(x) = x^2 - 10x + 23 = 0$

i.e. $x(x-10) = -23$

or $x = 10 - \dfrac{23}{x}$

Assume initial guess $x_0 = 5$.

i	x_i	$g(x_i)$	$\lvert f(x_i)\rvert < 1\times10^{-3}$
0	5	5.4	$\lvert -1.84\rvert$
1	5.4	5.7407	$\lvert -1.45\rvert$
2	5.7407	5.99352	$\lvert -1.0129\rvert$
3	5.99352	6.16252	
4	6.16252	6.26776	
5	6.26776	6.33043	
6	6.33043	6.36676	
7	6.36676	6.38749	
		6.39921	
		6.40581	
		6.40951	
		6.41158	
		6.41274	
		6.41339	

Please continue the iteration

To identify convergence of fixed point method

If you recall yesterday's problem on Archimedes' principle.

i.e. $\rho_f \frac{h^3}{3} - \rho_f R h^2 + \frac{4}{3} R^3 \rho_0 = 0$

i.e. If $\rho_f = 1.05$ g/cc
$\rho_0 = 1.50$ g/cc
$R = 1$ cm

Then.

$$\frac{1.05}{3} h^3 - 1.05 \times 1 \times h^2 + \frac{4}{3} 1^3 \times 1.50 = 0$$

i.e. $0.35 h^2 (h - 3) = -6$

i.e. $h - 3 = \frac{-17.14286}{h^2}$

or $h = 3 - \frac{17.14286}{h^2}$

This is of the form $x = g(x)$.

If you do iteration, you may see that you are not getting a proper solution.

∴ Convergence of the form $x_{i+1} = g(x_i)$ can be discussed as follows:

If $x = \alpha$ is the exact solution, then

$$x_{i+1} - \alpha = e_{i+1} \quad (\text{Error in } (i+1)^{\text{th}} \text{ step}).$$

$$= g(x_i) - g(\alpha)$$

Recall Taylor's series

$$f(x) = f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots$$

The same logic is applied here:

This Taylor's series of $g(\alpha)$ w.r.t x_i

$$g(\alpha) = g(x_i) + g'(x_i)(\alpha - x_i) + \frac{1}{2!} g''(x_i)(\alpha - x_i)^2 + \cdots$$

$\therefore$ Truncating after the first order diff. term.

$$g(\alpha) - g(x_i) = g'(x_i)(\alpha - x_i)$$

$$\text{or} \quad -e_{i+1} = g'(x_i)(-e_i)$$

$$\text{or} \quad e_{i+1} = g'(x_i)\, e_i$$

$$\text{or} \quad \left|\frac{e_{i+1}}{e_i}\right| = |g'(x_i)|$$

It any value ξ between x_i and α

i.e $\quad x_i \le \xi \le \alpha$

$$\left|\frac{e_{i+1}}{e_i}\right| = |g'(\xi)| < 1 \quad \text{for convergence.}$$

If $|g'(\xi)| > 1$, the procedure diverges

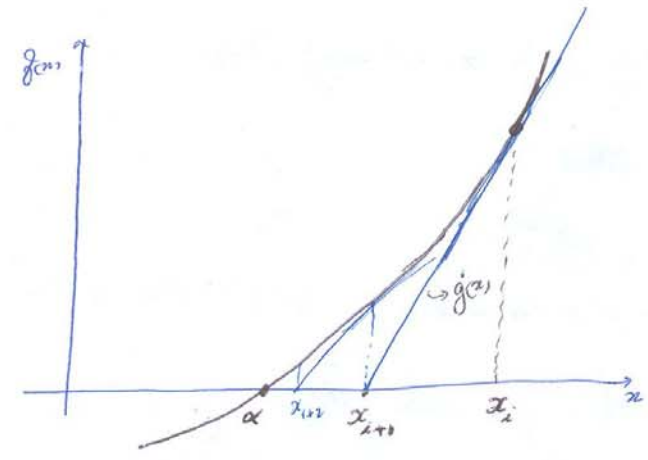
Newton's Method

→ Also called Newton Raphson method.

→ It is one of the most well-known method

→ Used in many engineering application and quite powerful.

(8)



→ This is a iterative procedure

→ Start with initial guess say x_i

→ Evaluate $f(x_i)$. Through the graphical point $(x_i, f(x_i))$ we draw a tangent line that intersects x-axis and we suggest the point as x_{i+1}

→ Again evaluate $f(x_{i+1})$.

→ Do the same procedure till $f(x_{i+1}) \to f(\alpha)$.

Slope of the straight line is

$$g'(x_i) = \frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

The slope of this straight line is also the slope of the curve at the point (x_i).

$\therefore g'(x_i) = f'(x_i)$.

$$\therefore f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

or $$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Repeat till convergence. $|x_{i+1} - x_i| \le \varepsilon_1$

or $|f(x_{i+1})| \le \varepsilon_2$, etc.

Using the Taylor series at point x_i

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{1}{2!} f''(x_i)(x_{i+1} - x_i)^2 + \cdots$$

Truncating after first order.

$$f(x_{i+1}) - f(x_i) = f'(x_i)(x_{i+1} - x_i)$$ and set $f(x_{i+1}) \approx 0.0$

or $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$