Lecture 14: Solution of Non-Linear Equation

(27-Aug-2012)

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27-AUG-2012

SOLUTION OF NON-LINEAR EQUATIONS

→ Till now we were discussing about solving systems of linear equations

→ Including Eigen value problems.

→ Though our discussions on eigen-value problems are not complete, we will continue them tomorrow

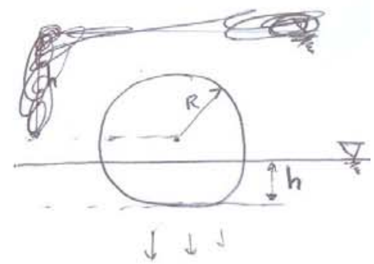
→ Today we will briefly introduce on non-linear equations solving methods.

Q. What is meant by non-linear equations?

S. Most of the features you see or observe may be not linear. They may have to be represented by non-linear equations

e.g. Consider our age-old experiment conducted by Archimedes:

* A spherical object of radius R
* To be sinked into a fluid (water or oil)
* Archimede's principle states that the object will sink to a depth 'h' from the surface at which the weight of fluid displaced by the sphere = Weight of the spherical object.



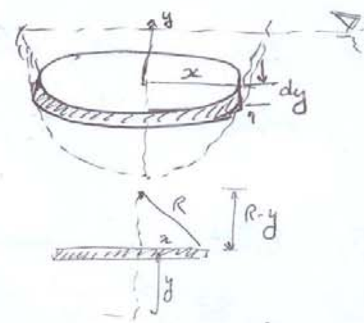
Weight of the spherical object $= \frac{4}{3}\pi R^3 \rho_o g$
(if ρ_o is its density)

Weight of the fluid displaced:

$\rho_f \rightarrow$ density of fluid

$V_d \rightarrow$ Volume of fluid displaced by the object

Then weight of fluid displaced $= \rho_f V_d g$



$\rightarrow$ How to determine V_d ?

$$V_d = \pi \int_0^h x^2 dy = \pi \int_0^h (2Ry - y^2)\, dy$$

$$= \pi \left[Rh^2 - \left(\frac{h^3}{3}\right) \right]$$

$$= \pi h^2 \left(R - \frac{h}{3} \right)$$

$$x^2 = R^2 - (R-y)^2$$
$$= R^2 - R^2 + 2Ry - y^2$$

$$\therefore \quad \frac{4}{3}\pi R^3 \rho_o g = \pi h^2 \left(R - \frac{h}{3}\right) \rho_f g$$

i.e. $$\rho_f \frac{h^3}{3} - \rho_f R h^2 + \frac{4}{3} R^3 \rho_o = 0$$

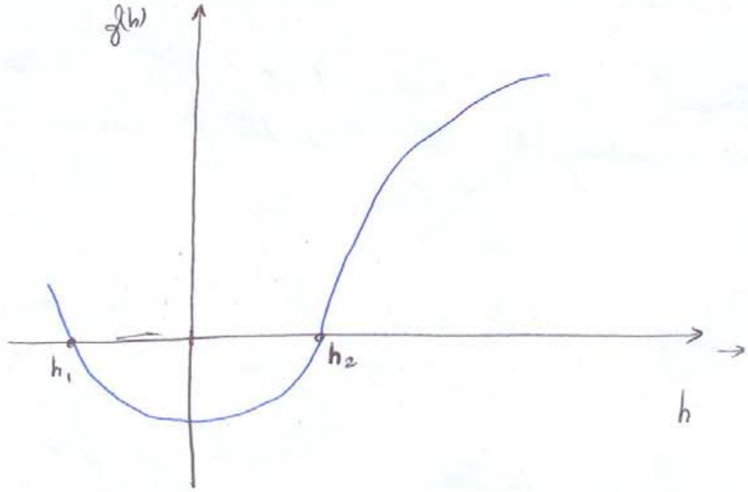
This is a non-linear equation in h.

i.e. $f(h) = 0$

On solving this equation, you will get the value of h that satisfies the Archimede's principle.

$\rightarrow$ To find the value of h means to find the roots of the equation $f(h) = 0$.

That is you can draw the plot h vs. $f(h)$ and find the roots.



→ In this case h_1 and h_2 are the roots of the equation $f(h) = 0$ as is evident from the figure.

Q: How will you find roots of such non-linear equations?

Sol. Most of engineering problems require solution of such non-linear equations.

→ The non-linear equation $f(h) = 0$ can be

* Algebraic
* Trigonometric, logarithmic, etc.
* Differential, etc.

→ There are two phases to find the roots.

* Bound the root
* Refine the root to accuracy.

Two types of root finding methods:

* Closed domain (i.e. Bracketing) methods
* Open domain (i.e. Non Bracketing) methods.

→ In <u>closed domain</u> in a closed interval the root is bracketed and subsequently the width of interval is shrunk.

As being discussing several times, the course is on Numerical methods → an approximate way to find solutions.

Therefore for non-linear equation $f(h) = 0$ or $f(x) = 0$ we can use our approximate methods. How?

<u>Bounding the Solution</u> :→ We will give (or find) a rough estimate for a solution of $f(x) = 0$
The estimate should be reasonable.

This can be done by
1) First graphing the function
or 2) Incremental search method. etc.
3) Experience
4) Solution of simplified cases. etc.

<u>Graphing</u>, helps in determining the range.

<u>Incremental Search</u> :→ You can increment the value of x and see where approximately $f(x) = 0$. or change sign.

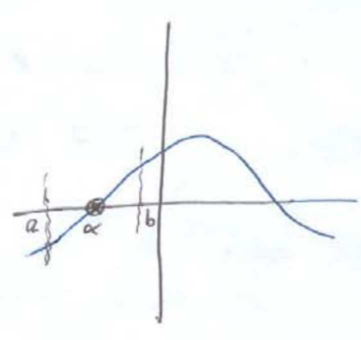
Refining the Solution

You can refine the solution by
→ Trial and error
→ Closed domain
→ Open domain

Closed Domain Methods

1) Method of Bisection :→
→ This is also called half interval or Bolzano method.
→ A simplest method.
→ The interval is halved in this method.

The method:
* You need to initially assume a suitable interval 'a' and 'b' where the solution of $f(x)=0$ may exist.
* To obtain this interval $[a,b]$ you may graph the function or substitute a in $f(x)$ and b in $f(x)$



i.e. Obtain $f(a)$ and $f(b)$ such that $f(x)=0$ lies within 'a' and 'b'.

i.e. The requirement is $f(x)$ should change sign from 'a' to 'b'.

i.e. $f(a)\,f(b) < 0$

* The root 'α' lies in between 'a' and 'b'.
* Now half this interval ~~b~~ between a and b such that $c = \frac{a+b}{2}$.

→ There are now two intervals $[a, c]$ and $[c, b]$.

Now check the product

$f(a) f(c)$
$f(c) f(b)$
} Based on which one is negative you will have the solution in that interval.

If $f(a) f(c) < 0$ then α lies in between $[a, c]$
If $f(c) f(b) < 0$ then " " " " $[c, b]$

* Now the process goes on till the solution is converged.

Instead of writing ~~a~~ 'a', 'c' and 'b' . you can write

x_i, $x_{i+1/2}$, and x_{i+1}.

Example.
Find the roots of the equation $f(x) = 2 - x^2$

Soln. Find a suitable initial bracket where we can ~~[illegible]~~ infer the solution exists.

If $x = 0$, $f(0) = 2$
If $x = 2$, $f(2) = -2$

∴ Definitely we can say solution exist between $[0, 2]$.

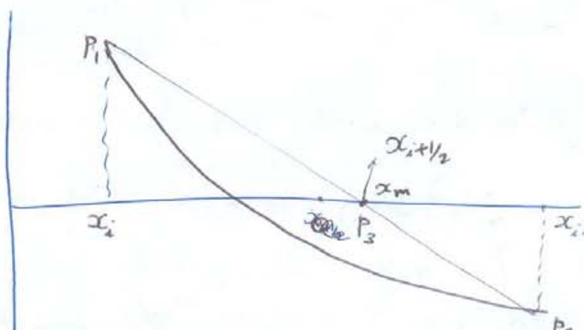
x_i	x_{i+1}	$f(x_i)$	$f(x_{i+1})$	$x_{i+1/2}$	$f(x_{i+1/2})$
0	2	2.00000	-2.00000	1.00000	1.00000
1.00000	2	1.00000	-2.00000	1.50000	-0.25000
1.00000	1.50000	1.00000	-0.25000	1.25000	0.44000
1.25000	1.50000	0.44000	-0.25000	1.37500	0.10938
1.37500	1.50000	0.10938	-0.25000	1.43750	-0.06641
1.37500	1.43750	0.10938	-0.06641	1.40625	0.02246
1.40625	1.43750	0.02246	-0.06641	1.42188	-0.02174

Continue the iteration

Method of false Position

To help converge faster, you require certain acceleration

→ In the Regula falsi method



Select two points of 'x' such that

$f(x_i)\, f(x_{i+1}) < 0$.

$f(x_i)$ is say the point P_1

At $f(x_{i+1})$ the point P_2.

Connect P_1 and P_2 by the straight line.
This straight line passes through P_3, where the ordinate is zero.
→ The slope of the straight line b/w P_1 and P_2

$$m = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

→ The same slope exist b/w P_3 and P_1

$$\text{i.e.} \quad m = \frac{0 - f(x_i)}{x_m - x_i} \qquad \left(\because \underset{\text{at point } P_3}{f(P_3)} = 0\right)$$

i.e. $\cancel{x_m}$

$$\therefore \quad \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{-f(x_i)}{x_m - x_i}$$

$$\text{or} \quad x_m - x_i = -f(x_i)\,\frac{(x_{i+1} - x_i)}{f(x_{i+1}) - f(x_i)}$$

$$\text{i.e.} \quad x_m = x_i - f(x_i)\,\frac{(x_{i+1} - x_i)}{f(x_{i+1}) - f(x_i)}$$

~~Ater~~ Evaluate the value of function $f(x_m)$ on the curve $f(x)$. If it is non-zero then
Again the possibilities are:

(i) If the product $f(x_i)\,f(x_m) < 0$. That mean the solution lies between $[x_i, x_m]$ closed interval.

(iii) If the product $f(x_m)\,f(x_{i+1}) < 0$. That mean that the solution lie between the interval $[x_m, x_{i+1}]$.

(iii) Do the process iteratively by replacing x_m as x_i or x_{i+1} appropriately in the next iteration.