Lecture 13: Fadeev-Leverrier Method, Similarity Transformation

(23-Aug-2012)

LECTURE – 13

23-AUGUST-2012

EIGEN VALUES & EIGEN VECTORS (Contd...)

If you look into last classes' lectures, we were dealing with Eigen problems.

→ We have seen what is meant by Eigen problem

$$[A]\{x\} = \lambda\{x\}$$

→ How to formulate an Eigen problem

→ Use power method, inverse power method, or shift power methods to find Eigen values.

To clarify some of your doubts:

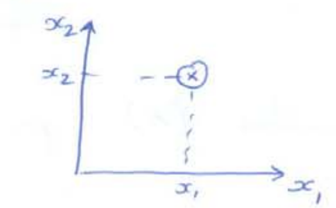
Recall System of Non-Homogeneous linear equations

$$[A]\{x\} = \{b\}$$

→ We suggested that there can be a unique solution for the above system if $[A]$ is non-singular.

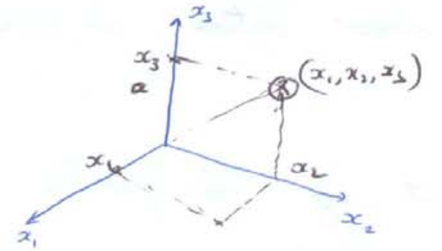
→ The meaning of unique solution is:

* For a two-dimensional plane the axes consists of (x_1, x_2).



The unique solution is a fixed point in the (x_1, x_2) plane.

* For a three-dimensional space represented by (x_1, x_2, x_3) co-ordinate system



In the three-dimensional space, the unique solution $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$, is a fixed point.

* Similarly in an n-dimensional ~~space~~ representation the unique solution,

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}$$ is a fixed point in the n-dimensional coordinate system.

→ In the Eigen-value problems:

$$[A]\{x\} = \lambda\{x\} \longrightarrow ①$$

Transformation of $\{x\}$ to a scalar product of its own is occurring.

Trivial solution $\{x\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$ already exist.

But we are not interested.

→ Therefore, for non-singular matrix $[A]$ you can define & obtain certain scalar values called Eigen values that can do the above transformation ①.

→ There can be more than one possibilities of λ. For $n \times n$ matrix $[A]$ there are n values of λ based on the characteristic polynomial.

$$\det(A - \lambda I) = 0$$

$$\text{i.e. } p(\lambda) = (-1)^n \left[\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots \right.$$
$$\left. \cdots + c_1\lambda + c_0 \right] = 0$$

→ That means that Eigen problems are not having unique solutions.

$$[A]\{x\} = \lambda\{x\}$$

Depending on the value of λ, you can have the corresponding Eigen vector.

$\lambda_{(j)}, \{x\}_{(j)}$ → are Eigen pairs.

→ In the power method we can find the dominant Eigen value and its corresponding Eigen vector.

Fadeev – Leverrier Method

As suggested earlier, ~~the method describes~~

$$\det(A - \lambda I) = 0$$

is
$$p(\lambda) = (-1)^n\left[\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots \right.$$
$$\left. \cdots + c_1\lambda + c_0\right] = 0$$

If we are able to find these coefficients $c_{n-1}, c_{n-2}, \ldots, c_1, c_0$ the characteristic polynomial can be solved.

To determine the coefficients of the polynomial we can use Fadeev – Leverrier Method.

→ Let $[A]$ be $n \times n$ matrix.

Now the trace of matrix $[A]$ is

$$Tr[A] = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Similarly we need to find traces of various matrices as follows:

⇒ Let us define matrix $[B]$, such that

$$[B_1] = [A]$$

Now define $p_1 = Tr[B_1]$

⇒ Again define
$$[B_2] = [A]\left([B_1] - p_1[I]\right)$$

and $p_2 = \frac{1}{2} Tr[B_2]$

$$\Rightarrow \quad [B_3] = [A]\left([B_2] - p_2[I]\right)$$

and $$p_3 = \frac{1}{3} Tr[B_3]$$

⋮

Similarly

$$[B_k] = [A]\left([B_{k-1}] - p_{k-1}[I]\right)$$

and $$p_k = \frac{1}{k} Tr[B_k]$$

⋮

$$[B_n] = [A]\left([B_{n-1}] - p_{n-1}[I]\right)$$

and $$p_n = \frac{1}{n} Tr[B_n]$$

Using these informations on traces, we define the characteristic polynomial

$$p(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} - \cdots$$
$$\cdots - p_{n-1}\lambda - p_n$$

Using this method you can also find inverse of $[A]$

$$[A]^{-1} = \frac{1}{p_n}\left([B_{n-1}] - p_{n-1}[I]\right)$$

Example: As taken from [http://ktuce.ktu.edu.tr/~pehlivan/numerical_analysis/chap08/FaddeevLeverrier.pdf]

Find characteristic polynomial of the matrix

$$[A] = \begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix}$$

Solution..

Using the Faddeev Leverrier method:

Assign $[B_1] = [A]$

$$\therefore [B_1] = \begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix}$$

$$\therefore p_1 = \text{Tr}[B_1] = 8+6+9+7 = 30$$

$$[B_2] = [A]\left([B_1] - p_1[I]\right)$$

$$= \begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix} \left(\begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 30 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -165 & 22 & -42 & 18 \\ 22 & -139 & -33 & 3 \\ -42 & -33 & -175 & -17 \\ 18 & 3 & -17 & -159 \end{bmatrix}$$

I have used Matlab for matrix multiplications

$$\therefore p_2 = \frac{1}{2}\text{Tr}[B_2] = -\frac{638}{2} = -319$$

Now $[B_3] = [A]\left([B_2] - p_2[I]\right)$

$$= \begin{bmatrix} 3618 & -425 & 1103 & -389 \\ -425 & 2906 & 770 & -34 \\ 1103 & 770 & 3958 & 386 \\ -389 & -34 & 386 & 3318 \end{bmatrix}$$

$\therefore$ ~~$p_3 = \frac{1}{3} Tr[B_3]$~~

$$= \begin{bmatrix} 1066 & -106 & 146 & -70 \\ -106 & 992 & 132 & -34 \\ 146 & 132 & 1087 & 67 \\ -70 & -34 & 67 & 1085 \end{bmatrix}$$

and $p_3 = \frac{1}{3} Tr[B_3] = 1410$

$$[B_4] = [A]\left([B_3] - p_3[I]\right)$$

$$= \begin{bmatrix} -2138 & 0 & 0 & 0 \\ 0 & -2318 & 0 & 0 \\ 0 & 0 & -2138 & 0 \\ 0 & 0 & 0 & -2138 \end{bmatrix}$$

and $p_4 = \frac{1}{4} Tr(B_4) = -2138$

$\therefore$ The characteristic polynomial will be:

$$p(\lambda) = \lambda^4 - 30\lambda^3 + 319\lambda^2 - 1410\lambda + 2138 = 0$$

Similarity Transformation

Recall the mathematics identity

Two matrices $[A]$ and $[B]$ of dimensions $n \times n$ are said to be similar if there exists an invertible matrix $[P]$ of $n \times n$ such that:

$$[B] = [P]^{-1}[A][P]$$

This is similarity transformation of $[A]$.

i.e. $$[B - \lambda I] = [P]^{-1}[A - \lambda I][P]$$

$$\det(B - \lambda I) = 0 = \det\left([P]^{-1}[A - \lambda I][P]\right)$$
$$= \det[P]^{-1} \cdot \det[A - \lambda I] \det[P]$$

As we are talking of $[P]$ as non-singular $\det[P]$ as well as $\det(P)^{-1}$ are not zero.

$$\therefore \det(B - \lambda I) = \det(A - \lambda I) = 0$$

That is the characteristic value (or Eigen value) of $[B]$ and $[A]$ are same.

→ Some matrices can be diagonalised using similarity transform.

i.e. Say $[D]$ is diagonal matrix

$$[D] = [P]^{-1}[A][P]$$

→ The property is the Eigen values of $[D]$ and $[A]$ will be same.

If you look into diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \\ \vdots & & & & & d_{nn} \\ 0 & 0 & \cdots & & & \end{bmatrix}$$

the Eigen values of the matrix will be the diagonal elements.

$\det(D - \lambda I) = 0$ will give $(d_{11} - \lambda)(d_{22} - \lambda) \ldots (d_{nn} - \lambda) = 0$

or $\lambda = d_{11}, d_{22}, d_{33}, \ldots, d_{nn}$

$\therefore$ If you transform $[A]$ to $[D]$ you will get the elements of (D) will be Eigen values of $[A]$.

→ In a same way note that ~~the~~ for any triangular matrix $[L]$ or $[U]$, the

$$[U - \lambda I] = \begin{bmatrix} u_{11} - \lambda & u_{12} & \cdots & u_{1n} \\ 0 & (u_{22} - \lambda) & \cdots & u_{2n} \\ 0 & 0 & \cdots & u_{3n} \\ \vdots & & & (u_{nn} - \lambda) \\ 0 & 0 & \cdots & \end{bmatrix}$$

the characteristic polynomial is

$(u_{11} - \lambda)(u_{22} - \lambda) \ldots (u_{nn} - \lambda) = 0$

or $\lambda_i = u_{ii}$ $\quad (i = 1, 2, 3, \ldots, n)$