Lecture 12: Power Method, Inverse Power Method, Shifted Power Method

(22-Aug-2012)

Power Method

* This is an iterative procedure.
* It is based on repetitive multiplication of a trial eigen vector. $\{x\}^{(0)}$ with a scaling factor.
* The method computes the dominant eigen pair λ, $\{x\}$

Note: $\rightarrow$ If for $n \times n$ non-singular matrix $[A]$.

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \dots \quad |\lambda_n|$$

$\lambda_1 \rightarrow$ Dominant eigen value $\quad \{x\}_{(1)} \rightarrow$ Dominant eigen vector.

We have to normalise the eigen vector in the process.

$$\{x\}_{(1)} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}_{(1)} \quad \rightarrow \quad \text{Find the max value the} \quad \frac{1}{c}\{x\}_{(1)}$$

Procedure:

$[A]\{x\} = \lambda\{x\}$ is the eigen problem.

Assume $\{x\}^{(0)}_{(1)}$

Assign $\{y\}^{(1)} = [A]\{x\}^{(0)}$

Now $\{x\}^{(1)} = \frac{1}{c^{(1)}}\{y\}^{(1)}$

Next Step $\{y\}^{(2)} = [A]\{x\}^{(1)}$

$\{x\}^{(2)} = \frac{1}{c^{(2)}}\{y\}^{(2)}$

Next Step $\{y\}^{(3)} = [A]\{x\}^{(2)}$

$\{x\}^{(3)} = \frac{1}{c^{(3)}}\{y\}^{(3)}$

The process continues till a convergence is reached.

The convergence criteria is $\left| \dots \right| \leq \varepsilon$

On convergence. $c^{(k)} = \lambda_1$

Example

Compute the largest Eigen value and corresponding eigen vector of

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Solution:

We need to find the largest Eigen value λ_1

such that: $|\lambda_1| > |\lambda_2| > |\lambda_3| > |\lambda_4|$

Let us initially assume $\{x\}^{(0)}_{(1)} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

First Iteration

$\therefore \{y\}^{(1)} = [A]\{x\}^{(0)}$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{Bmatrix}$$

```
>> A = [2 -1 0 0; -1 4 -1 0; 0 -1 4 -1; 0 0 -1 2]; y1 = [2; -1; 0; 0]; x1 = (1/2)*y1
x1 =
   1.00000000000000
  -0.50000000000000
                  0
                  0
```

```
>> y2=A*x1
y2 =
   2.50000000000000
  -3.00000000000000
   0.50000000000000
                  0

>> x2 = (1/3.0)*y2
x2 =
   0.83333333333333
  -1.00000000000000
   0.16666666666667
                  0

>> y3=A*x2
y3 =
   2.66666666666667
  -5.00000000000000
   1.66666666666667
  -0.16666666666667

>> x3 = (1/5)*y3
x3 =
   0.53333333333333
  -1.00000000000000
   0.33333333333333
  -0.03333333333333

>> y4=A*x3
y4 =
   2.06666666666667
  -4.86666666666667
   2.36666666666667
  -0.40000000000000

>> x4 = (1/4.86666666666667)*y4
x4 =
   0.42465753424658
  -1.00000000000000
   0.48630136986301
  -0.08219178082192

>> y5=A*x4
y5 =
   1.84931506849315
```

```
  -4.91095890410959
   3.02739726027397
  -0.65068493150685

>> x5 = (1/4.91095890410959)*y5
x5 =
   0.37656903765690
  -1.00000000000000
   0.61645746164575
  -0.13249651324965

>> y6=A*x5
y6 =
   1.75313807531381
  -4.99302649930265
   3.59832635983263
  -0.88145048814505

>> x6 = (1/4.99302649930265)*y6
x6 =
   0.35111731843575
  -1.00000000000000
   0.72067039106145
  -0.17653631284916

>> y7=A*x6
y7 =
   1.70223463687151
  -5.07178770949720
   4.05921787709497
  -1.07374301675978

>> x7 = (1/5.07178770949720)*y7
x7 =
   0.33562813240073
  -1.00000000000000
   0.80035248113675
  -0.21170898276147

>> y8=A*x7
y8 =
   1.67125626480146
  -5.13598061353748
   4.41311890730848
  -1.22377044665969
```

```
>> x8 = (1/5.13598061353748)*y8
x8 =
   0.32540159135265
  -1.00000000000000
   0.85925536706200
  -0.23827396143865

>> y9=A*x8
y9 =
   1.65080318270530
  -5.18465695841465
   4.67529542968666
  -1.33580328993931

>> x9 = (1/5.18465695841465)*y9
x9 =
   0.31840162154336
  -1.00000000000000
   0.90175598254359
  -0.25764545285321

>> y10=A*x9
y10 =
   1.63680324308673
  -5.22015760408696
   4.86466938302758
  -1.41704688825002

>> x10 = (1/5.22015760408696)*y10
x10 =
   0.31355437272722
  -1.00000000000000
   0.93190086429937
  -0.27145672520320

>> y11=A*x10
y11 =
   1.62710874545444
  -5.24545523702659
   4.99906018240068
  -1.47481431470577

>> x11 = (1/5.24545523702659)*y11
x11 =
   0.31019400069779
  -1.00000000000000
```

```
   0.95302694551911
  -0.28116040420960

>> y12=A*x11
y12 =
   1.62038800139558
  -5.26322094621690
   5.09326818628605
  -1.51534775393832

>> x12 = (1/5.26322094621690)*y12
x12 =
   0.30787003204953
  -1.00000000000000
   0.96770936244791
  -0.28791262411804

>> y13=A*x12
y13 =
   1.61574006409907
  -5.27557939449745
   5.15875007390968
  -1.54353461068398

>> x13 = (1/5.27557939449745)*y13
x13 =
   0.30626779416576
  -1.00000000000000
   0.97785469389208
  -0.29258105987258

>> y14=A*x13
y14 =
   1.61253558833152
  -5.28412248805784
   5.20399983544090
  -1.56301681363723

>> x14 = (1/5.28412248805784)*y14
x14 =
   0.30516620157384
  -1.00000000000000
   0.98483709399280
  -0.29579496258265

>> y15=A*x14
```

```
y15 =
   1.61033240314768
  -5.29000329556664
   5.23514333855387
  -1.57642701915811

>> x15 = (1/5.29000329556664)*y15
x15 =
   0.30441047257896
  -1.00000000000000
   0.98962950418977
  -0.29800114122410

>> y16=A*x15
y16 =
   1.60882094515791
  -5.29403997676873
   5.25651915798318
  -1.58563178663797

>> x16 = (1/5.29403997676873)*y16
x16 =
   0.30389285918084
  -1.00000000000000
   0.99291263025021
  -0.29951262053102

>> y17=A*x16
y17 =
   1.60778571836168
  -5.29680548943105
   5.27116314153185
  -1.59193787131224

>> x17 = (1/5.29680548943105)*y17
x17 =
   0.30353875020893
  -1.00000000000000
   0.99515890323887
  -0.30054678701884

>> y18=A*x17
y18 =
   1.60707750041785
  -5.29869765344780
   5.28118239997433
```

```
  -1.59625247727655

>> x18 = (1/5.29869765344780)*y18
x18 =
   0.30329669770309
  -1.00000000000000
   0.99669442292823
  -0.30125373849891

>> y19=A*x18
y19 =
   1.60659339540619
  -5.29999112063132
   5.28803143021183
  -1.59920189992605

>> x19 = (1/5.29999112063132)*y19
x19=
   0.30313133717379
  -1.00000000000000
   0.99774345085732
  -0.30173671304860

>> y20=A*x19
y20 =
   1.60626267434758
  -5.30087478803111
   5.29271051647787
  -1.60121687695451

>> x20 = (1/5.30087478803111)*y20
x20 =
   0.30301841461609
  -1.00000000000000
   0.99845982561752
  -0.30206653448407

>> y21=A*x20
y21 =
   1.60603682923218
  -5.30147824023361
   5.29590583695416
  -1.60259289458565

>> x21 = (1/5.30147824023361)*y21
x21 =
```

```
   0.30294132248695
  -1.00000000000000
   0.99894889632157
  -0.30229170468406

>> y22=A*x21
y22 =
   1.60588264497391
  -5.30189021880852
   5.29808728997032
  -1.60353230568968

>> x22 = (1/5.30189021880852)*y22
x22 =
   0.30288870170812
  -1.00000000000000
   0.99928272207057
  -0.30244539956733

>> abs((5.30189021880852-5.30147824023361)/5.30147824023361)
ans =
    7.771013220108145e-005
```

(This is greater than the criteria of $\varepsilon \leq 1\times10^{-5}$. So continue the iteration)

```
>> y23=A*x22
y23 =
   1.60577740341624
  -5.30217142377869
   5.29957628784962
  -1.60417352120524

>> x23 = (1/5.30217142377869)*y23
x23 =
   0.30285278899411
  -1.00000000000000
   0.99951055223952
  -0.30255029364215

>> y24=A*x23
y24 =
   1.60570557798822
  -5.30236334123363
   5.30059250260021
  -1.60461113952381

>> x24 = (1/5.30236334123363)*y24
```

```
x24 =
   0.30282828140077
  -1.00000000000000
   0.99966602842554
  -0.30262187561641

>> abs((5.30236334123363-5.30217142377869)/5.30217142377869)
ans =
    3.619601095502531e-005
```

(This is again greater than the criteria of $\varepsilon \leq 1\times10^{-5}$. So continue the iteration)

```
>> y25=A*x24
y25 =
   1.60565656280154
  -5.30249430982631
   5.30128598931858
  -1.60490977965836

>> x25 = (1/5.30249430982631)*y25
x25 =
   0.30281155791644
  -1.00000000000000
   0.99977212224340
  -0.30267072171756

>> y26=A*x25
y26 =
   1.60562311583287
  -5.30258368015983
   5.30175921069114
  -1.60511356567851

>> x26 = (1/5.30258368015983)*y26
x26 =
   0.30280014662295
  -1.00000000000000
   0.99984451551952
  -0.30270405192929

>> y27=A*x26
y27 =
   1.60560029324590
  -5.30264466214248
   5.30208211400738
  -1.60525261937810
```

```
>> x27 = (1/5.30264466214248)*y27
x27 =
   0.30279236033085
  -1.00000000000000
   0.99989391177970
  -0.30272679420490

>> y28=A*x27
y28 =
   1.60558472066170
  -5.30268627211055
   5.30230244132369
  -1.60534750018949

>> x28 = (1/5.30268627211055)*y28
x28 =
   0.30278704759628
  -1.00000000000000
   0.99992761578431
  -0.30274231169074

>> abs((5.30268627211055-5.30264466214248)/5.30264466214248)
ans =
    7.847021763765108e-006
```

$7.847021763765108e-006 \leq 1\times10^{-5}$.

The solution is converged. The dominant eigen value is = 5.30269.

The corresponding Eigen vector is:

0.30279
-1.00000
0.99993
-0.30274

Inverse Power Method

For an $n \times n$ matrix $[A]$ with unique eigen value of min. absolute magnitude, it can be obtained by the power method.

If λ_i's for $[A]$ - eigen value

Then $\frac{1}{\lambda_i}$ are eigen values for $[A]^{-1}$.

$\therefore$ If we apply power method on $[A]^{-1}$, we will get ~~mean~~ max eigen value ~~of~~ for $[A]^{-1}$ - i.e. minimum for $[A]$.

i.e. $$[A]\{x\} = \lambda\{x\}$$

$$[A]^{-1}[A]\{x\} = \lambda[A]^{-1}\{x\}$$

$$\text{or } [A]^{-1}\{x\} = \frac{1}{\lambda}\{x\}$$

We can use the Doolittle (LU) method to solve such case:

$$\text{Say } [A]^{-1}\{x\}^{(k)} = \{y\}^{(k+1)}$$

$$\text{or } [A][A]^{-1}\{x\}^{(k)} = [A]\{y\}^{(k+1)}$$

i.e. Your system is

$$[A]\{y\}^{(k+1)} = \{x\}^{(k)}$$

You know or start guess with $\{x\}^{(k)}$

Then you find $\{y\}^{(k+1)}$.

$\rightarrow$ You required to solve that system.

As it is iterative technique - we factorise [A] to [L] and [U] . (maybe we Doolittle method).

$$\therefore \quad [L]\{z\} = \{x\}^{(k)} \rightarrow \text{Forward Substitution}$$

$$[U]\{y\}^{(k+1)} = \{z\} \rightarrow \text{Backward Substitution}$$

The Algorithm is :

(i) Obtain [L] and [U] matrices for [A] using Doolittle algorithm.

(ii) Assume $\{x\}^{(0)}$ such that max magnitude of ~~the~~ any component is one.

(iii) Solve for $\{z\}$

i.e. $[L]\{z\} = \{x\}^{(k)}$

(iv) Solve for $\{y\}^{(1)}$ from

$[U]\{y\}^{(1)} = \{z\}$

(v) Assign $\{x\}^{(1)} = \frac{1}{\lambda}\{y\}^{(1)}$

(vi) Again repeat the step to find $\{z\}$, subsequently $\{y\}^{(2)}$ and $\{x\}^{(2)}$

(vii) Repeat till convergence of the $\frac{1}{\lambda}$ value.

Shifted Power Method

* If you have a scalar magnitude 's' and you want to find the eigen value of the system
$$[A]\{x\} = \lambda\{x\}$$
closest to 's' then we need to use this method.

The scalar quantity is 's'

$$\therefore \; s[I]\{x\} = s\{x\}$$

$$[A]\{x\} - s\{x\} = \lambda\{x\} - s\{x\}$$

i.e. $$[A - sI]\{x\} = (\lambda - s)\{x\}$$

→ Now $[A - sI]$ is a shifted matrix and using same power method you can find the eigen value of the shifted matrix i.e $= \lambda - s$.

Quiz - 4

1) What is the maximum magnitude of any components for Eigen vectors possible if you use power method.
2) At a time how many eigen values can be evaluated using power method.
3) In three or four steps prove inverse power method gives the min. eigen value of [A].
4) In shifted power method, the coefficient matrix and eigen value for the power method are ____ and ____ .