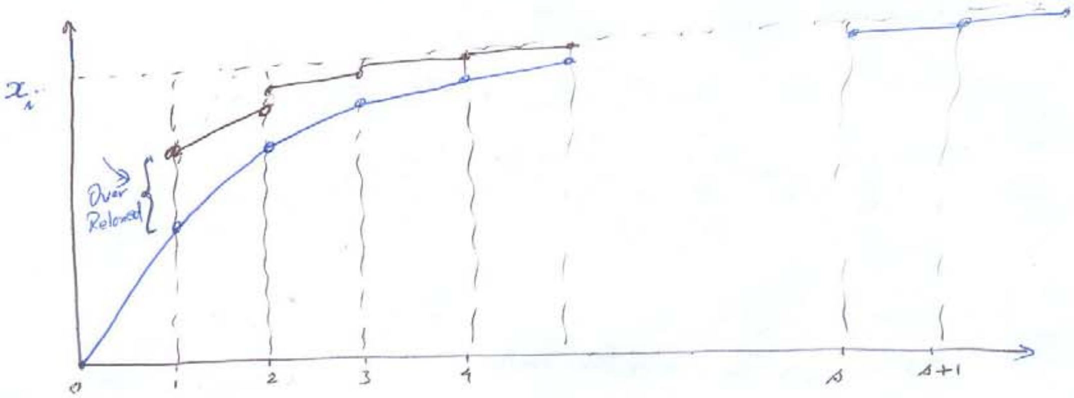
Lecture 10: **(14-Aug-2012)**

Successive - Over Relaxation

If you look into the iteration process closely you see that ~~iteration~~

$$\{x\}^{(0)} \rightarrow \{x\}^{(1)} \rightarrow \{x\}^{(2)} \rightarrow \cdots \rightarrow \{x\}^{(s+1)}$$



→ As discussed earlier asymptotically the ~~solution~~ approximation reach the exact solution as iteration increases.

→ The entire procedure is called relaxation

→ The initial vector $\{x\}^{(0)}$ is successively relaxed to $\{x\}_{exact}$

→ Now the direction of relaxation can be seen in the above figure.

→ What happens if in the first iteration, I over relax the solution vector by a certain amount.

→ You may see that on doing it successively, you may converge faster.

∴ SORM (Successive Over-Relaxation Method) work on this principle.

→ The Gauss-Seidel Iteration is modified.

$$x_i^{(s+1)} = x_i^{(s)} + \omega \frac{R_i^{(s)}}{a_{ii}}$$

$$i = 1, 2, 3, \ldots, n$$

$$R_i^{(s)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(s)}$$

$$i = 1, 2, 3, \ldots, n$$

ω = Over-relaxation factor

$1.0 < \omega < 2.0$

The Same Example

$$\begin{bmatrix} 5 & 0 & -2 \\ 3 & 5 & 1 \\ 0 & -3 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 2 \\ -4 \end{Bmatrix}$$

Take $\omega = 1.10$

Solution.

Take $\{x\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$$\therefore \; x_i^{(s+1)} = x_i^{(s)} + \omega \frac{R_i^{(s)}}{a_{ii}} \quad ; \; i = 1, 2, 3$$

$$R_i^{(s)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=1}^{n} a_{ij} x_j^{(s)}$$

For $s = 1$

$R_1^{(0)} = 7$, $x_1^{(1)} = 0 + 1.1 \times \frac{7.0}{5.0} = 1.54$

$R_2^{(0)} = 2 - [3 \times 1.54 + 0 + 0] = -2.62$; $x_2^{(1)} = 0 + 1.1 \times \frac{-2.62}{5} = -0.5764$

$R_3^{(0)} = -4 - [0 + (-3 \times -0.5764) + 0] = -5.7292$; $x_3^{(1)} = 0 + 1.1 \times \frac{-5.7292}{4} = -1.5755$

Continue the iteration.

EIGEN VALUES AND EIGEN VECTORS

Recall our linear system representation

$$[A]\{x\} = \{b\} \rightarrow (1)$$

$\{x\}$ is a vector having n-components. So naturally we can say it is a vector in n-dimensional space.

→ This vector $\{x\}$ is transformed to another n-component vector $\{b\}$ in the same n-dimensional space.

→ When you studied stress analysis, you may be recalling the term principal stress (or principal direction) etc.

→ In a two-dimensional plane you can find the principal directions based on a relation $[A - \lambda I] = 0$

→ In 2D case there exist two independent vectors say $\{\vec{X}_1\}$ and $\{\vec{X}_2\}$ that satisfies the above relation.

→ Similarly in n-dimensional space there exist n-independent vectors $\{\vec{X}_1\}, \{\vec{X}_2\}, \{\vec{X}_3\}, \dots \{\vec{X}_n\}$

→ Each of them consist of n-components.

$\therefore$ In the relation

$$[A]\{\vec{X}\} = \{\vec{b}\}$$

we can find certain vector $\{\vec{b}\}$ such that it is a scalar product of a quantity λ and the vector $\{\vec{X}\}$.

i.e. $$[A]\{\vec{X}\} = \lambda\{\vec{X}\}$$

We want to find vector $\{\vec{X}\}$ and scalar λ such that

$$[A]\{\vec{X}\} = \lambda\{\vec{X}\}$$

i.e. This linear transformation maps $\{\vec{X}\}$ onto its multiple $\lambda\{\vec{X}\}$.

Now you can see that

$$[A]\{\vec{X}\} - \lambda\{\vec{X}\} = 0$$

or $$[A - \lambda I]\{\vec{X}\} = 0 \rightarrow (2)$$

$\rightarrow$ This linear system has non-trivial solutions iff $[A - \lambda I]$ is singular matrix.

i.e. $$\det(A - \lambda I) = 0$$

i.e. $$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ a_{31} & \cdots & & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

This determinant i.e. $\det(A - \lambda I) = 0$ is a polynomial of degree n. This is called characteristic polynomial.

$$p(\lambda) = \det(A - \lambda I)$$
$$= (-1)^n \left(\lambda^n + c_1 \lambda^{n-1} + \cdots c_{n-1} \lambda + c_n \right)$$

There can be n roots for this polynomial.

How Eigen value problem formulated ?