CE 601: Numerical Methods
Lecture 9

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

Solution Of Non-Linear Equations

- Q. What is non-linear equation?

- Most of the engineering and scientific problems may not be linear.
- E.g. Let us demonstrate the age old experiment conducted by Archimedes.

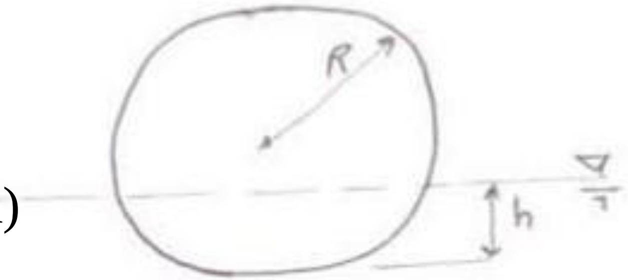
➢ The spherical object of radius *R* immersed into a fluid (water or oil).

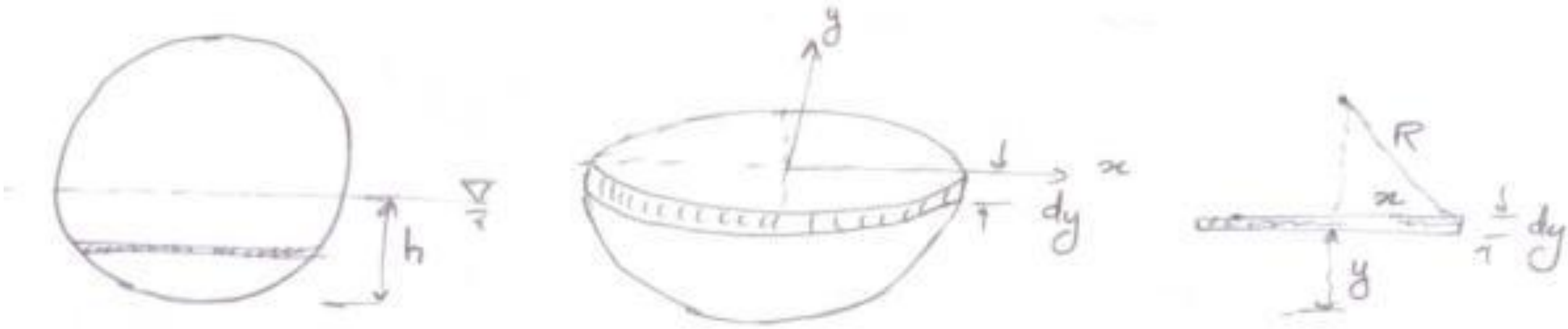
➢ The object will sink to a depth *h* from the surface of water.

➢ At this situation, weight of the spherical object = weight of fluid displaced by the sphere

$$\text{Weight of sphere} = \frac{4}{3}\pi R^3 \rho_s g \ \ (\rho_s \rightarrow \text{density of sphere})$$

$$\text{Weight of fluid} = \rho_f V_d g \ \ (V_d \rightarrow \text{volume of fluid displaced})$$





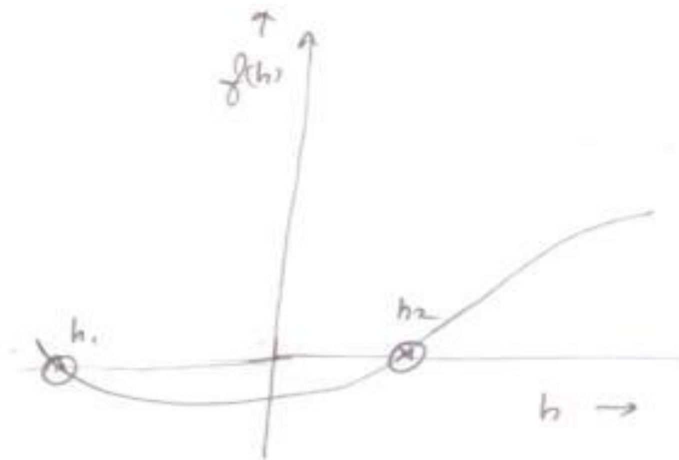
$$V_d = \pi \int_0^h x^2 dy = \pi \int_0^h \left(2Ry - y^2\right) dy = \pi h^2 \left(R - \frac{h}{3}\right)$$

$$\therefore \frac{4}{3} \pi R^3 \rho_s g = \pi h^2 \left(R - \frac{h}{3}\right) \rho_f g$$

$$\Rightarrow \rho_f \frac{h^3}{3} - \rho_f R h^2 + \frac{4}{3} R^3 \rho_s = 0 \cdots (1)$$

- On solving eq. (1) you can get the value of *h* to interpret Archimedes’s principle.

- Eq. (1) is $f(h) = 0$, a non-linear equation of variable h.
- You need to find the root of $f(h) = 0$. h_1, h_2 etc can be the roots of $f(h) = 0$.



- Q. How to find the roots of Non-Linear equation

➢ The non-linear equations of form $f(h) = 0$ can be

❖ Algebraic

❖ Trigonometric, logarithmic etc.

❖ Differential etc.

- ✓ Two steps are involved to find roots of $f(h) = 0$ by iterative methods
  - ❖ Bound the root
  - ❖ Refine the roots
- ✓ There are -> Closed domain (Bracketing methods)
  -> Open domain (Non-bracketing methods)

  to solve the non-linear $f(h) = 0$.

- <u>Closed domain</u>: The root is bracketed initially within two values and the width of this values are shrunk till $f(h) = 0$.
- Q. How to bound the solution for $f(h) = 0$.

➢ Provide an initial estimate of an interval *'a', 'b'* within which the true solution *h* exists.

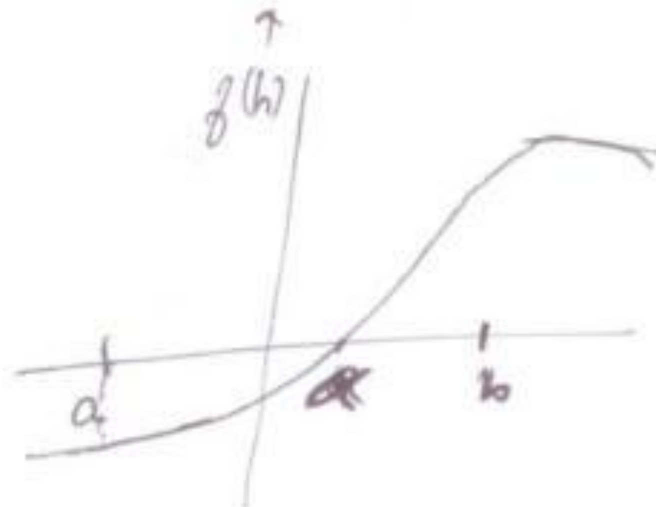
Closed Domain Method

1) <u>Method of Bisection</u>

- In this method what you do is initially assumed as interval between *'a'* and *'b'* such that there exists the value α which gives $f(\alpha) = 0$.

- Steps:

▪ Assume a suitable interval between 'a' and 'b' such that $f(x) = 0$ exist in between.



▪ To obtain this [a, b] you may graph the function or evaluate $f(a)$ and $f(b)$ such that $f(a).f(b) < 0$, then solution lies in between *'a'* and *'b'*.

- Half the interval between [*a,b*] by incorporating $c = (a+b)/2$. Two intervals [*a,c*] and [*c,b*] exist now.
- Check *f(a).f(c)* and *f(c).f(b)* and the solution exists in that interval whose product is negative i.e. [*a,c*] or [*c,b*].
- Proceed the method with solution interval till solution converges.

- <u>Example:</u> Find the roots of the equation $f(x) = 2 - x^2$.

➢ Initially find a suitable bracket [a,b] such that $f(a).f(b) < 0$.

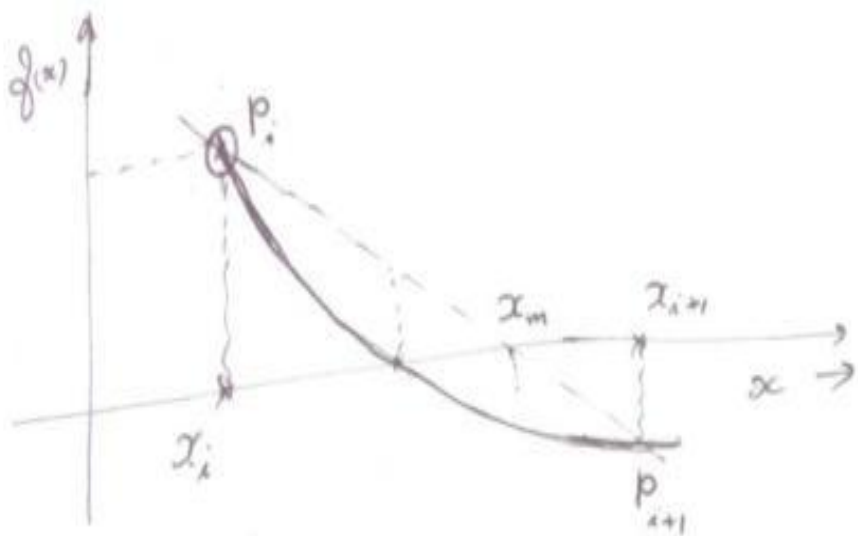
➢ Let $x = 0$, $f(0) = 2$ and $x = 2$, $f(2) = -2$. So solution lies between [0,2].

x_i	x_{i+1}	$f(x_i)$	$f(x_{i+1})$	$x_{i+1/2}$	$f(x_{i+1/2})$
0	2	2.00000	-2.00000	1.00000	1.00000
1.00000	2.00000	1.00000	-2.00000	1.50000	-0.25000
1.00000	1.50000	1.00000	-0.25000	1.25000	0.44000
1.25000	1.50000	0.44000	-0.25000	1.37500	0.10938
1.37500	1.50000	0.10938	-0.25000	1.43750	-0.06641
1.37500	1.43750	0.10938	-0.06641	1.40625	0.02246
1.40625	1.43750	0.02246	-0.06641	1.42188	-0.02174

- Calculate this iteration till you observe $f(x_{i+1/2}) \approx 0.00$.
- In reality $f(x_{i+1/2}) \approx 0.00$ may not be possible. We may fix a convergence criteria $| f(x_{i+1/2}) | \leq 1 \text{ X } 10^{-4}$ etc.

- <u>Method of False position (Regula Falsi Method)</u>

- Unlike bisection method where the interval [*a,b*] is halved at every iteration, in regula falsi method as such:



- Bracketed between $[x_i, x_{i+1}]$ identify the two points $P_i\ (x_i, f(x_i))$ *and* $P_{i+1}\ (x_{i+1}, f(x_{i+1}))$.
- Connect a straight line between P_i and P_{i+1}. The line cuts the x-axis at x_m.
- Now we can check in which interval of $[x_i, x_m]$ and $[x_m, x_{i+1}]$ solution lies.

- We determine the slope of the straight line between P_i and P_{i+1} as such

$$m = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

- Slope between P_i and P_m will be same m

$$\therefore m = \frac{0 - f(x_i)}{x_m - x_i}$$

$$\text{i.e.,} \frac{-f(x_i)}{x_m - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\Rightarrow x_m = x_i - \frac{f(x_i)\left(x_{i+1} - x_i\right)}{f(x_{i+1}) - f(x_i)}$$

- We have to select the bracket among $[x_i, x_m]$ and $[x_m, x_{i+1}]$ whichever product of function is negative i.e. $f(x_i).f(x_m) < 0$ or $f(x_m).f(x_{i+1}) < 0$

- Example: Solve $f(x) = x^2 - 10x + 23$.

➢ Initially let $x_i = 2.0000$, $f(x_i) = 7$ and $x_{i+1} = 4.0000$, $f(x_{i+1}) = -1$.

➢ Solution lies between [2,4]. Use convergence criteria $|f(x_m)| \leq 1 \times 10^{-4}$.

$$x_m = x_i - \frac{f(x_i)\left(x_{i+1} - x_i\right)}{f(x_{i+1}) - f(x_i)}$$

x_i	x_{i+1}	$f(x_i)$	$f(x_{i+1})$	x_m	$f(x_m)$
2.0000	4.0000	7.0000	-1.0000	3.7500	-0.4375
2.0000	3.7500	7.0000	-0.4375	3.6471	-0.1696
2.0000	3.6471	7.0000	-0.1697	3.6081	-0.0627
2.0000	3.6081	7.0000	-0.0626	3.5938	-0.0227
2.0000	3.5938	7.0000	-0.0226	3.5887	-0.0081
2.0000	3.5887	7.0000	-0.0082	3.5868	-0.0029
2.0000	3.5868	7.0000	-0.0029	3.5862	-0.0012
2.0000	3.5862	7.0000	-0.0012	3.5859	-0.0003
2.0000	3.5859	7.0000	-0.0003	3.5858	-0.0001

Now $|f(x_m)| \leq 1 \text{ X } 10^{-4}$

- So, one solution of $f(x) = x2 – 10x + 23$

  is <u>$x = 3.5858$</u>.

- From these two methods you can see that closed domain method guarantees convergence. However they may be slow.
- Alternative to overcome the slowness is to use open domain methods.
- <u>Open Domain Methods:</u>

- The roots are not bracketed as like in closed domain methods.

- Fixed-point iteration
- Newton's method
- Secant method
- Muller's method etc

- <u>Fixed Point Iteration</u>

- To solve $f(x) = 0.$
- Rearrange $f(x)$ in such a way that $x = g(x)$
  - → Provide initial guess for x say x_i
  - → Evaluate $g(x_i)$
  - → If not equal then, $x_{i+1} = g(x_i)$
  - → Evaluate $g(x_{i+1})$
  - → Continue till some tolerance ε i.e., $|x_{i+1} – x_i| \le \varepsilon$

- <u>Example:</u> Solve $f(x) = f(x) = x^2 -10x + 23.$
  - $f(x) = x^2 -10x + 23.$

Or, $x = 10 – (23/x) = g(x)$

Assume initial guess

$x_o = 3.0$ and $\varepsilon = 1 \text{ X } 10^{-4}$

i	x_i	$g(x_i)$	$\lvert x_{i+1} – x_i \rvert \leq \varepsilon$
0	**3.000**	**2.333333**	
1	**2.333333**	**0.1428557**	
2	**0.1428557**	**-151.0016**	
3	**-151.0016**	**10.15232**	
4	**10.15232**	**7.734508**	

- Up to *21* iterations required to reach $x_i = 6.414295.$