CE 601: Numerical Methods
Lecture 8

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Iterative Methods to solve systems of Linear equations

- Many of the engineering (and scientific) problems involve system of large number of equations.
- You have already seen the elimination methods to solve such systems.
- In Gauss elimination, the number of computations involved for an *(n X n)* system
  - ✓ n steps in back substitution
  - ✓ At each k^{th} step,

    (n-k) computations for l_{ik}

    2(n-k)(n-k+1) for a_{ij}'s

i.e. $$\sum_{k=1}^{n-1} (n-k) + 2(n-k)(n-k+1) \ + n$$

$$= \frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6} + n \approx \frac{2n^3}{3}$$

- You can imagine the number of computations that may be required for 1000 x 1000 system.
- In most cases you have extremely sparse [A].
- They may be diagonally dominant.
- The diagonally dominant matrices can be solved by iterative methods.

- Note: A matrix [A] is said to be diagonally dominant if

$$\left|a_{ij}\right| \geq \sum_{j=1,\, j\neq i}^{n} \left|a_{ij}\right|; i = 1, 2, \ldots, n$$

- Common iterative methods are:
  - ✓ Jacobi iteration
  - ✓ Gauss-Seidel iteration
  - ✓ Successive over relaxation , etc.
- Q. What is the process adopted in any iterative scheme?
- Usual steps involved in solving [A]{x} = {b}
  - o Assume initial solution vector $\{x\}^{(0)}$.
  - o Using this initial guess improve to get $\{x\}^{(1)}$ and then using $\{x\}^{(1)}$ get $\{x\}^{(2)}$.
  - o This process goes on till $\{x\}^{(s)}$ converges to actual solution.
  - o Diagonal dominant matrices are used.

Jacobi Method

- As mentioned earlier, the iterative techniques work for diagonally dominant matrix.

$$[A]\{x\} = \{b\}$$

$$\text{or, } \sum_{j=1}^{n} a_{ij} x_j = b_i; i = 1, 2, 3, \ldots, n$$

$$[A] = [L] + [D] + [U],$$

where $[L] \rightarrow$ a lower triangular matrix with zeroes on the diagonal,

$[U] \rightarrow$ a upper triangular matrix with zeroes on the diagonal,

$[D] \rightarrow$ a diagobal matrix.

$$\therefore [A]\{x\} = \{b\} \Rightarrow \left[[L] + [D] + [U] \right] \{x\} = \{b\}$$

$$\Rightarrow [D]\{x\} = \{b\} - \left[[L] + [U] \right] \{x\} \qquad \rightarrow (1)$$

Now objective is to give initial guess $x^{(0)}$

and utilise in the equation (1) to get

$$[D]\{x\}^{(1)} = \{b\} - \left[[L] + [U] \right] \{x\}^{(0)}$$

$$\vdots \qquad \qquad \vdots$$

The process goes on till defined tolerance meets.

- In simple form,

$$[A]\{x\} = \{b\}$$

$$\text{or, } \sum_{j=1}^{n} a_{ij} x_j = b_i; i = 1, 2, 3, ..., n$$

for any i^{th} row,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ii}x_i + \ldots + a_{in}x_n = b_i$$

Now from this equation you have,

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j \right); i = 1, 2, 3, ..., n$$

Now if you have initial vector $\{x\}^{(0)}$,then improved solution vector$\{\mathrm{x}\}^{(1)}$ has the components,

$$x_i^{(1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(0)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(0)} \right); i = 1, 2, 3, ..., n$$

The procedure is repeated.

For, say (s+1)th iteration

$$x_i^{(s+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(s)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(s)}\right); i = 1,2,3,\ldots,n$$

$$\Rightarrow \{x\}^{(s+1)} = [D]^{-1}\ \{b\} - ([L]+[U])\{x\}^{(s)}$$

We also can write this as,

$$x_i^{(s+1)} = x_i^{(s)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(s)} - a_{ii}x_i^{(s)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(s)}\right)$$

$$= x_i^{(s)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{n} a_{ij}x_j^{(s)}\right)$$

- You know the system [A]{x} = {b}. For an exact solution {x}, you have {b} – [A]{x} = 0 i.e. ,

  $$b_i - \sum_{j=1}^{n} a_{ij}x_j = 0 \text{ (theoretically)}$$

- However in the iterative techniques our initial guess $\{x\}^{(0)} \neq \{x\}$ actual solution.

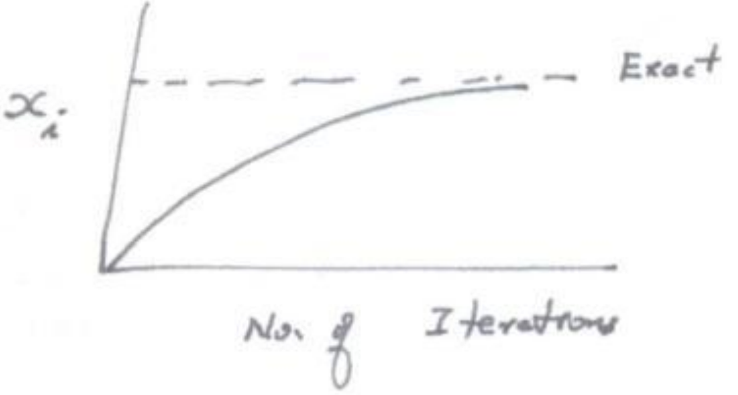
- We can formulate a residual quantity for each row at any iteration, i.e.,

$$R_i^{(s)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(s)}$$

$$\therefore x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{ii}}; i = 1, 2, 3, \ldots, n$$

You proceed the iteration till $R_i^{(s)} \approx 0.00$

- Iterative methods approach the exact solution asymptotically as the number of iterations are increased.



- Absolute error

  = Approximate value (from s[th] iteration) - Exact value

- Relative error = (Absolute error/Exact value)
- The iterative procedure needs to converge to the solution or exact value (Convergence)
- Recall, $x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{ii}}$

$$\text{i.e., } \Delta x_i = x_{i+1}^{(s+1)} - x_i^{(s)}$$

$$\left|\Delta x_i\right| = \frac{R_i^{(s)}}{a_{ii}} \text{ is the residual criteria.}$$

- For a system there will be n residuals in each iteration. How will you check the convergence?

- $\left| \Delta x_{i\ \max} \right| \le \varepsilon$ is the criteria, we may adopt.

  Or, we may incorporate

  $$\left| \frac{\Delta x_{i\ \max}}{x_i} \right| \le \varepsilon \quad \text{(Relative criteria for convergence)}$$

  Or, it can be $\sum_{i=1}^{n} \left| \frac{\Delta x_i}{x_i} \right| \le \varepsilon$

  Or, it can be $\left[\sum_{i=1}^{n} \left(\frac{\Delta x_i}{x_i} \right)^2 \right]^{1/2} \le \varepsilon$

  $\varepsilon \rightarrow$ Some chosen numerical value for lower limit.

- <u>Gauss-Siedel Iteration</u>

$$x_i^{(s+1)} = x_i^{(s)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{n} a_{ij} x_j^{(s)} \right); i = 1, 2, 3, \ldots, n$$

- If you are sequentially proceeding from first equation to the final n^th equation, then after any i^th equation you are aware of $x_1^{(s+1)}, x_2^{(s+1)}, \ldots, x_{i-1}^{(s+1)}$.

$$\therefore\ x_i^{(s+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(s)} \right); i = 1, 2, 3, \ldots, n$$

Improved knowledge is used earlier here,

$$x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{ii}}; i = 1, 2, 3, \ldots, n$$

$$R_i^{(s)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(s)}; i = 1, 2, 3, \ldots, n$$

- <u>Example of Iterative Methods</u>

Solve using Jacobi's iteration the system,

$$\begin{pmatrix} 5 & 0 & -2 \\ 3 & 5 & 1 \\ 0 & -3 & 4 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 2 \\ -4 \end{Bmatrix}.$$

Soln. At each iteration $x_i^{(s+1)} = x_i^{(s)} + \dfrac{R_i^{(s)}}{a_{ii}}$

We have x_1, x_2, x_3 and $R_i^{(s)} = b_i - \sum_{j=1}^{3} a_{ij} x_j^{(s)}$

Begin at $s = 0, \{x\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$

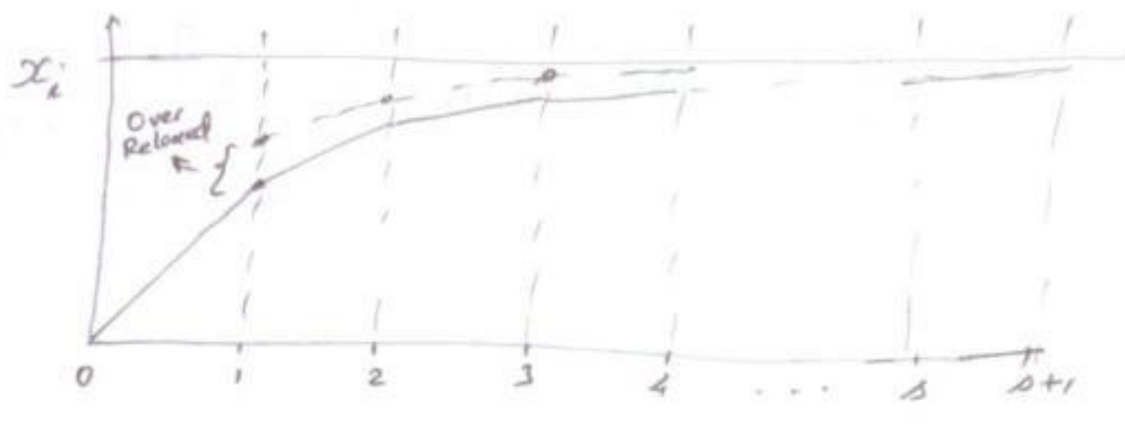
Iterate the method.

s	$x_1^{(s)}$	$x_2^{(s)}$	$x_3^{(s)}$	$R_1^{(s)}$	$R_2^{(s)}$	$R_3^{(s)}$
0	0	0	0	7	2	-4
1	1.40	0.40	-1.00	-2	-3.2	1.2
2	1.00	-0.24	-0.700	0.6	0.9	-1.92
3	1.12	-0.06	-1.18			

- You have to give a good initial guess.
- <u>Successive Over Relaxation</u>

o The mechanism of iterative methods:

$\{x\}^{(0)} \rightarrow \{x\}^{(1)} \rightarrow \ldots \rightarrow \{x\}^{(s+1)}$



- The iterative process is called relaxation .
- The initial vector $\{x\}^{(0)}$ is successively relaxed to $\{x\}_{exact}$.
- The direction of relaxation if it is same as figure shows, then we can think of over relaxing at each iteration.
- This may increase the speed of convergence.
- The Gauss-Seidel iteration is modified as such

$$x_i^{(s+1)} = x_i^{(s)} + w\frac{R_i^{(s)}}{a_{ii}}; i = 1, 2, 3, \ldots, n$$

$$w \rightarrow \text{over relaxation factor, } 1.0 < w < 2.0$$

$$R_i^{(s)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(s)}$$

- <u>Example</u>

$$\begin{bmatrix} 5 & 0 & -2 \\ 3 & 5 & 1 \\ 0 & -3 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 2 \\ -4 \end{Bmatrix} \quad \text{Take } w = 1.10$$

$$\{x\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ (The initial guess)}$$

$$R_1^{(0)} = 7,$$

$$x_1^{(1)} = 0 + 1.1 \times \frac{7.0}{5.0} = 1.54,$$

$$R_2^{(0)} = 2 - (3 \times 1.54 + 0 + 0) = -2.62$$

s	$x_1^{(s)}$	$x_2^{(s)}$	$x_3^{(s)}$	$R_1^{(s)}$	$R_2^{(s)}$	$R_3^{(s)}$
0	0	0	0	7	-2.62	-5.7292
1	1.54	-0.576	-1.5755			

Convergence of Iterative Methods

Recall, $x_i^{(s+1)} = x_i^{(s)} + \dfrac{R_i^{(s)}}{a_{ii}}$

Also recall earlier for Jacobi iteration, we write:

$$\{x\}^{(s+1)} = [D]^{-1}\{b\} - [D]^{-1}([L]+[U])\{x\}^{(s)}$$

$$\Rightarrow \{x\}^{(s+1)} = [S]\{x\}^{(s)} + \{c\} \qquad \rightarrow (1)$$

where for Jacobi iteration

$$[S] = -[D]^{-1}([L]+[U])$$

$$\{c\} = [D]^{-1}\{b\}$$

For Gauss-Seidel,

$$[S] = -[D]^{-1}([L]+[U])$$

$$\{c\} = ([L]+[U])^{-1}\{b\}$$

- The form of iteration in (1) remains unchanged. These are stationary iteration methods. [*S*] -> iteration matrix.

- Now if $\{x\}$ is true solution, then error vector in any s^{th} iteration:

  $\{e\}^{(s)} = \{x\} - \{x\}^{(s)}$

  Also (1) is applicable to true solution:

  i.e. $\{x\} = [S]\{x\} + \{c\}$

  So, $\{e\}^{(s+1)} = \{x\} - \{x\}^{(s+1)} = [S]\{x\} + \{c\} - [S]\{x\}^{(s)} - \{c\}$

  $\therefore \{e\}^{(s+1)} = [S]\{e\}^{(s)}$

- Error vector satisfies homogeneous form of iteration.
- At start $\{e\}^{(0)}$ -> known vector from $\{x\}(0)$
- To satisfy convergence,

  $$\lim_{s\to\infty}\{e\}^{(s)} = 0.$$

  This happens when $\lim_{s\to\infty}[S]^{(s)} = 0.$

Let us assume from $[A]$ that

$[S] \rightarrow$ has a set of lineraly independent eigen vectors and eogen values.

$\lambda_j; \ j = 1, 2, 3, \ldots, n \rightarrow$ the eigen values

$$\{e\}^{(0)} = \sum_{j=1}^{n} C_j \{v_j\}; \quad \{v_j\} \rightarrow \text{ the eigen vector}$$

Then $\{e\}^{(s)} = \sum_{j=1}^{n} C_j \lambda_j^s \{v_j\}$

The largest eigen value should be strictly less than 1 for convergence.

This is Spectral radius,

$\rho(s) < 1.0 \quad \rightarrow$ necessary condition to converge.