CE 601: Numerical Methods
Lecture 7

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Drawback in Elimination Methods

There are various drawbacks while using elimination methods using computers

- Theoretically, the elimination methods should give actual solution.
- However in computers we declare or assign variables with only certain precision (say single precision, double precision, etc.)

Due to this there are drawbacks like:

- Accumulation of round-off errors.
- Failure in solving ill-conditioned systems.

- <u>Round-off Errors</u>

- Round-off errors are generated by approximating precision numbers by finite precision numbers.
- Usually in computer you have single precision numbers (7 significant digits) and double precision numbers (14 significant digits).
- The case of effect of round-off errors in elimination method is demonstrated in next slide.

$$\begin{pmatrix} 0.0003 & 3 \\ 1 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1.0002 \\ 1 \end{Bmatrix}$$

Using Gauss elimination method,

$$\begin{pmatrix} 0.0003 & 3 & \vdots 1.0002 \\ 1 & 1 & \vdots \quad 1 \end{pmatrix} (R_2 = R_2 - (1/0.0003)R_1)$$

$$\Rightarrow \begin{pmatrix} 0.0003 & 3 & \vdots 1.0002 \\ 0 & -9999 & \vdots -3333 \end{pmatrix}$$

$$\therefore x_2 = 0.3333333........$$

$$x_1 = 0.666666.........$$

- But while using computer, you need to use finite precision numbers.

Accuracy of significant Digits	Precision	X_2	x_1
4	4 decimals	0.3333	1.000
5	5 decimals	0.33333	0.70000
6	6 decimals	0.333333	0.670000
7	7 decimals	0.3333333	0.6670000

- You can see from the above table, how based on the significant digits and precision the round-off errors creep.
- You can reduce round-off errors by doing partial pivoting.

- <u>Ill-conditioned System</u>

Consider the system

$$x_1 + x_2 = 2$$

$$x_1 + 1.0001x_2 = 2.0001$$

$$\text{or,} \begin{bmatrix} 1 & 1 & \vdots & 2 \\ 1 & 1.0001 & \vdots & 2.0001 \end{bmatrix}$$

It's solution is $x_2 = 1.0$ and $x_1 = 1.0$.

- if we give a slight perturbation on the coefficient of x_2 in the above equation,

$$i.e., x_1 + x_2 = 2$$

$$x_1 + 0.9999x_2 = 2.0001$$

$$\text{or,} \begin{bmatrix} 1 & 1 & \vdots & 2 \\ 1 & 0.9999 & \vdots & 2.0001 \end{bmatrix}$$

Now the solutions are $x_2 = -1$ and $x_1 = 3$.

- Huge difference in solution for a very slight changes in the system.
- This is because the system is ill-conditioned.

- An *ill-conditioned* system is one where small changes in the numerical values of coefficient matric [A] or right side vector{b} cause large changes in the solution vector {x}.
- A *well-conditioned* system is one which produce only small change in the solution vector for small changes in [A] or {b}.

  Example of an ill-conditioned system has already been discussed in our previous class.
- Ill-conditioning of a system is determined by its condition number.

- <u>Norms</u>

- Measure of magnitude of a matrix, vector, etc. For [*A*], {*b*}, {*x*} etc, their norms are given as :

  $\|A\|, \|b\|, \|x\|$ etc.

✓ The norm of a matrix or vector shall be always greater than zero.

$$\text{i.e., } \|A\| > 0, \|x\| > 0$$

$$\text{If } \|A\| = 0, \text{then } [A] = 0.$$

✓ If *k* is a scalar quantity, then *k*[*A*]=[*kA*], but

$$\|kA\| = |k|\|A\|, \text{(norm of a scalar is its absolute value)}$$

- ✓ For [A] and [B],

$$\|A+B\| \le \|A\| + \|B\|$$

$$\|AB\| \le \|A\|\|B\|$$

- ✓ Norm of a vector $\{x\} \rightarrow \|x\|$

You have, $\|x\|_1 = \sum |x_i|$ (sum of magnitudes)

$$\|x\|_2 = \|x\|_e = \left(\sum x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i| \quad \text{(Maximum magnitude Norm)}$$

- ✓ For an n × n matrix [A],

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \text{ (Maximum of column sums)}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \text{ (Maximum of row sums)}$$

$$\|A\|_e = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \text{ (Euclidean norm)}$$

- <u>Condition Number</u>

○ Measure of sensitivity of the system to small changes in any of its elements.

Recall, $[A]\{x\} = \{b\}$

Now it is clear, $\|b\| \leq \|A\|\|x\|$ (from properties of norm)

Small change $\{\delta b\}$ in $\{b\}$ will cause change in $\{x\}$,say by $\{\delta x\}$.

So, new system will be, $[A]\ \{x\} + \{\delta x\}\ = \{b\} + \{\delta b\}$

$\Rightarrow [A]\{\delta x\} = \{\delta b\}\ (\because [A]\{x\} = \{b\})$

$\Rightarrow \{\delta x\} = [A]^{-1}\{\delta b\}$

$\|\delta x\| \leq \left\|A^{-1}\right\|\|\delta b\|$

$\Rightarrow \|b\|\|\delta x\| \leq \|A\|\|x\|\left\|A^{-1}\right\|\|\delta b\|\ (\because \|b\| \leq \|A\|\|x\|)$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A\|\left\|A^{-1}\right\|\frac{\|\delta b\|}{\|b\|}$$

where $c(A) = \|A\|\left\|A^{-1}\right\|$ is called condition number.

i.e., $\dfrac{\|\delta x\|}{\|x\|} \leq c(A)\dfrac{\|\delta b\|}{\|b\|}$

✓ Smaller values of $c(A)$ -> well conditioning

✓ Larger values of $c(A)$ -> ill conditioning

- Example

$$\text{Find condition number of } [A] = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \end{pmatrix}$$

$$\|A\|_e = 2.00005$$

$$[A]^{-1} = 10000\begin{pmatrix} 1.0001 & -1 \\ -1 & 1.0001 \end{pmatrix} = \begin{pmatrix} 10001 & -10000 \\ -10000 & 10001 \end{pmatrix}$$

$$\|A^{-1}\|_e = 20000.50002$$

$$\therefore c(A) = \|A\|\|A^{-1}\| = 40002$$

$$c(A) = \text{Very large value} \rightarrow \text{Ill-conditioned system.}$$