CE 601: Numerical Methods

Lecture 41

Eigen Values

&

Eigen Vectors

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We have studied earlier about system of linear equations

i.e. $[A]\{x\} = \{b\}$

$\rightarrow$ A linear system will be homogeneous, say if $[B]\{x\} = 0$

$\rightarrow$ This homogeneous sytem, if we can represent

as $[A - \lambda I]\{x\} = 0 \quad \rightarrow$ (1A)

where $I \rightarrow$ unit matrix

$A \rightarrow$ another square matrix

$\lambda \rightarrow$ some scalar values

Then that means that $[B] = [A - \lambda I]$

or, $[A]\{x\} = \lambda\{x\} \quad \rightarrow$ (1B)

i.e. the matrix $[A]$ transforms the vector $\{x\}$ by simply multiplying with a scalar quantity.

Eq.(1) has trivial solution $\{x\} = 0$.

But this is of not interest to us.

Therefore, we need to make $[A - \lambda I]$ as singular

i.e. $\det(A - \lambda I) = 0$.

The parameters 'λ' that can make the matrix $[A - \lambda I]$ as singular are called eigen values.

Again in $[A]\{x\} = \lambda \{x\}$

→ For different values of λ, we get different vectors $\{x\}$

i.e. for $\lambda_1 \to \{x\}_1, \lambda_2 \to \{x\}_2, \ldots$

→ These vectors that are solutions of $[A]\{x\} = \lambda \{x\}$ are called eigen vactors.

$\rightarrow$ To find eigen values:

$$\det\left(A-\lambda I\right)=0$$

i.e. $$\begin{bmatrix} (a_{11}-\lambda) & a_{12} & \cdots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-\lambda) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = 0$$

determinant is some polynomial in λ

$$p(\lambda)=(-1)^n\left[\lambda^n+c_1\lambda^{n-1}+\cdots+c_{n-1}\lambda+c_n\right]=0$$

Solving this non-linear polynomial will give you

$\rightarrow$ The Eigen values.

Numerical Methods to Determine Eigen Values & Eigen Vectors

$\rightarrow$ The traditional method to solve non-linear equations as discussed in earlier lectures can be employed to solve non-linear polynomial

$$(-1)^n\left[\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots c_{n-1}\lambda + c_n\right] = 0$$

These are other methods as well.

I. Power Method

$*$ This is an iterative procedure.

$*$ It is based on repetitive multiplication of a trial eigen vector $\{x\}^{(0)}$ with a scaling factor.

$*$ We assume a trial eigen vector.

$*$ The normalisation is done on contents of $\{x\}$.

$*$ The procedure will determine the dominant eigen pair $\left[\lambda_{\max}, \{x\}\right]$.

$\rightarrow$ If for an $n \times n$ non-singular matrix $[A]$ if the eigen values $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$, then $\lambda_1 \rightarrow$ dominant eigen value and $\{x\}_1 \rightarrow$ eigen vector corresponding to domiant eigen value.

The procedure is

(i) Assume $\{x\}_1^{(0)}$

(ii) Assign $\{y\}^{(1)} = [A]\{x\}^{(0)}$

(iii) Identity $\lambda_1^{(1)} = \max\left|y_i^{(1)}\right|$

(iv) Evaluate $\{x\}^{(1)} = \dfrac{1}{\lambda_1^{(1)}}\{y\}^{(1)}$

(v) Again evaluate $\{y\}^{(2)} = [A]\{x\}^{(1)}$

Determine as above $\lambda_1^{(2)} = \max\left|y_i^{(2)}\right|$

$$\{x\}^{(2)} = \frac{1}{\lambda_1^{(2)}}\{y\}^{(2)}$$

(vi) Continue the process till convergence

i.e. $\dfrac{\left|\lambda_1^{(k+1)} - \lambda_1^{(k)}\right|}{\left|\lambda_1^{(k)}\right|} \le \varepsilon$ or as per requirement.

Example:

Compute the largest eigen value of $[A] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Answer. The matrix $[A]$ is 4×4.

$\therefore$ There are four eigen values for $[A]$. The homogeneous system is $[A - \lambda I]\{x\} = 0$. Let $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|$.

To find λ_1

Assign $\{x\}_1^{(0)} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

$$\{y\}^{(1)} = [A]\{x\}^{(0)} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. $\{y\}^{(1)} = \begin{Bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{Bmatrix}, \therefore \lambda_1^{(1)} = \max\left|y_i^{(i)}\right| = 2$

$$\{x\}^{(1)} = \frac{1}{\lambda_1^{(1)}}\{y\}^{(1)} = \frac{1}{2}\begin{Bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.5 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{y\}^{(2)} = [A]\{x\}^{(1)} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.5 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.5 \\ -3.0 \\ 0.5 \\ 0 \end{Bmatrix}$$

$\therefore \lambda_1^{(2)} = 3.0$

and $\{x\}^{(2)} = \frac{1}{3.0}\begin{Bmatrix} 2.5 \\ -3.0 \\ 0.5 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.83373 \\ -1.0000 \\ 0.16667 \\ 0.00000 \end{Bmatrix}$

The process goes on till there is no change in value of λ for successive iteration.

After many iterations,

$\lambda_1 = 5.30269$

$$\{x\}_1 = \begin{Bmatrix} 0.30279 \\ -1.0000 \\ 0.99993 \\ -0.30274 \end{Bmatrix}$$

II. Inverse Power Method

For an $n \times n$ matrix $[A]$ with unique eigen value of minimum absolute magnitude, an be obtained by power method

$[A] \rightarrow \lambda_i$ (Eigen values for matrix A)

for $[A]^{-1} \rightarrow \dfrac{1}{\lambda_i}$ (Eigen values of matrix A^{-1})

$\therefore$ Applying power method on $[A]^{-1}$ will give you the dominant eigen value of $[A]^{-1}$, which happens to be the minimum eigen value of $[A]$.

$$[A]\{x\} = \lambda\{x\}$$

$$[A]^{-1}[A]\{x\} = \lambda[A]^{-1}\{x\}$$

$$\therefore [A]^{-1}\{x\} = \frac{1}{\lambda}[I]\{x\} = \frac{1}{\lambda}\{x\}$$

III. Shifted Power Method

$$[A]\{x\} = \lambda\{x\}$$

If you have some scalar value $'s'$ and you are interested to find the closest eigen value of $[A]$ corresponding to $'s'$.

Then let, $s[I]\{x\} = s\{x\}$

$$[A]\{x\} - s\{x\} = \lambda\{x\} - s\{x\}$$

$$\Rightarrow [A - sI]\{x\} = (\lambda - s)\{x\}$$

$[A - sI] \rightarrow$ shifted matrix.

$$[A - sI]_{shifted}\{x\} = \lambda_{shifted}\{x\},$$

where, $\lambda_{shifted}$ is the shifted eigen value.

Apply power method to detrmine the dominant eigen value of the shifted system.