CE 601: Numerical Methods

Lecture 40

Galerkin FEM for Laplace Equation-2

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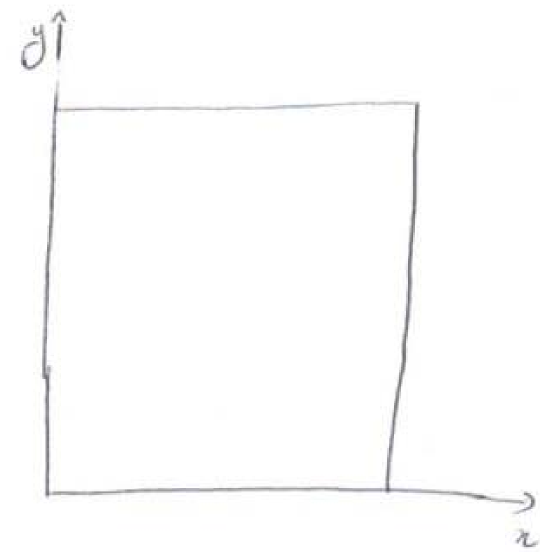
Galerkin FEM

We were discussing on how Galerkin FEM can be applied for 2-D cases.



$\rightarrow$ to solve 2-D Laplace equation (or Poisson equation).

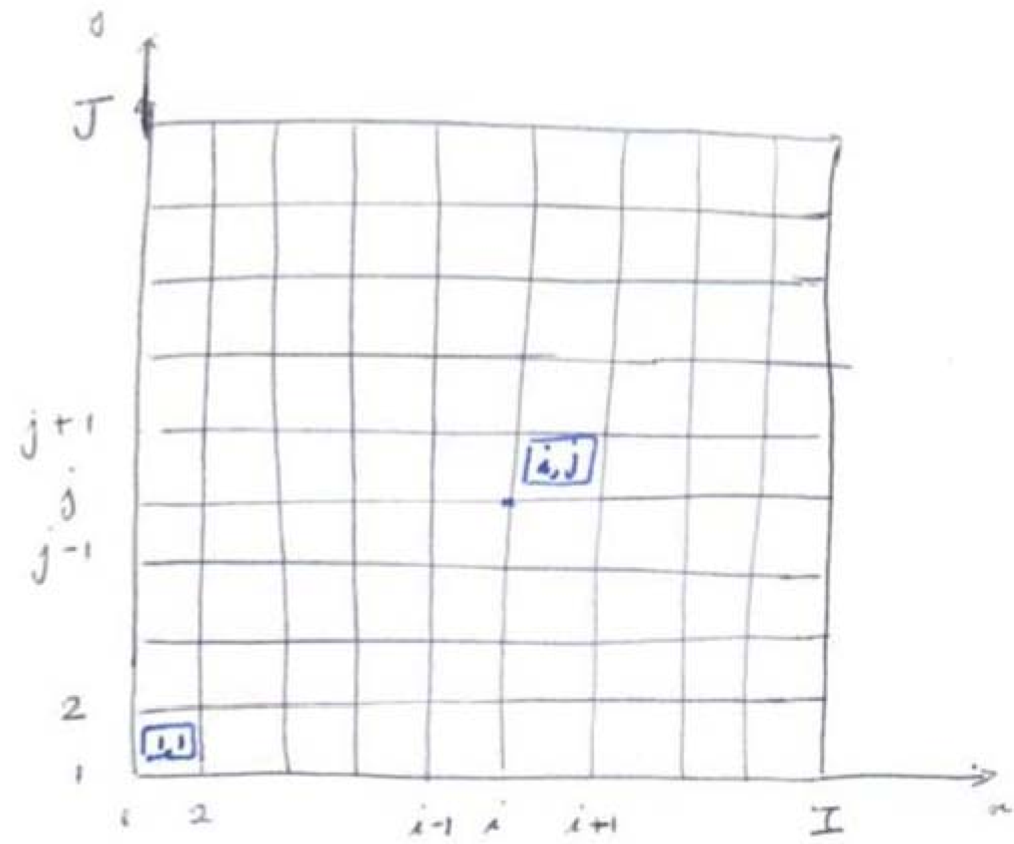
$$\text{i.e.} \quad \left.\begin{array}{c} \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = F(x, y) \\ \text{or} \\ \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0 \end{array}\right\} \text{with appropriate BCs.}$$

$\rightarrow$ two dimensional domain was rectangular in shape.

$\rightarrow$ In our case we discretised the domain using rectangular elements.

$\therefore$ The $x-$axis is discretised $i=1,2,3,\ldots,I$ nodes or discrete points

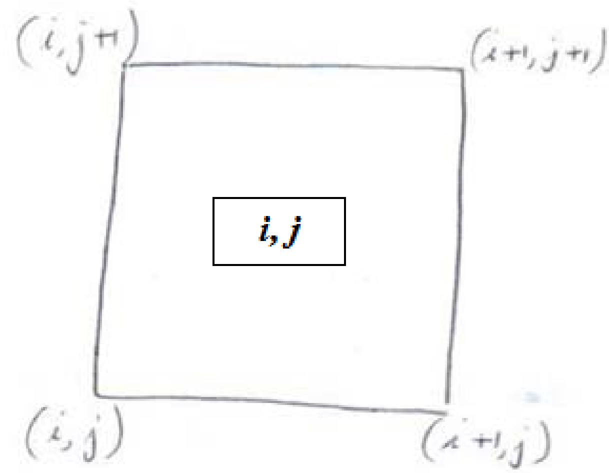
The $y-$axis is discretised $j=1,2,3,\ldots,J$ nodes.



$\therefore$ For this rectangular domain there are a total of $(I \times J)$ discrete nodes.

$\rightarrow$ Any general node is given as (i, j) in the suffix.

$\rightarrow$ Four nodes constitute an rectangular element. There are a total of $(I-1)\times(J-1)$ elements. Any general element is given as $\boxed{i,\, j}$ in the superfix.



$\rightarrow$ As described for the one-dimensional case, there will be combination of approximating polynomial for ' f ' in the entire domain

i.e. $f(x, y) \approx \tilde{f}(x, y) = \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \tilde{f}^{[i,j]}(x, y)$

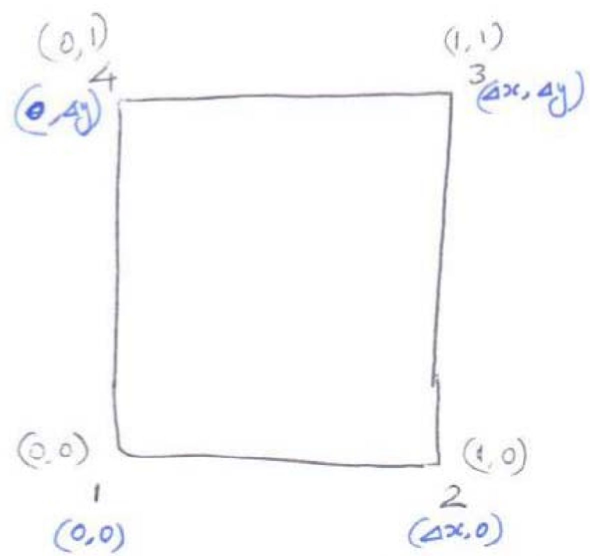
(i.e. Sum of series of local interpolating polynomials)

$\rightarrow$ For an element $\boxed{i, j} \rightarrow$ the approximation is $\tilde{f}^{[i,j]}(x, y)$

$\rightarrow$ As there are four nodes $(i, j), (i+1, j), (i+1, j+1)$ and $(i, j+1)$ associated with the element $\boxed{i, j}$, the approximation is given as:

$$\tilde{f}^{[i,j]}(x, y) = f_1 N_1(x, y) + f_2 N_2(x, y) + f_3 N_3(x, y) + f_4 N_4(x, y)$$

i.e. for the element $\boxed{i, j}$, the global node numbers are replaced by local node numbers $1, 2, 3, 4$ and N_1, N_2, N_3, N_4 are shape functions for the element $\boxed{i, j}$.



$\rightarrow$ We defined the <u>Residual</u> for the problem and evaluated weighted integral.

$$R(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - F$$

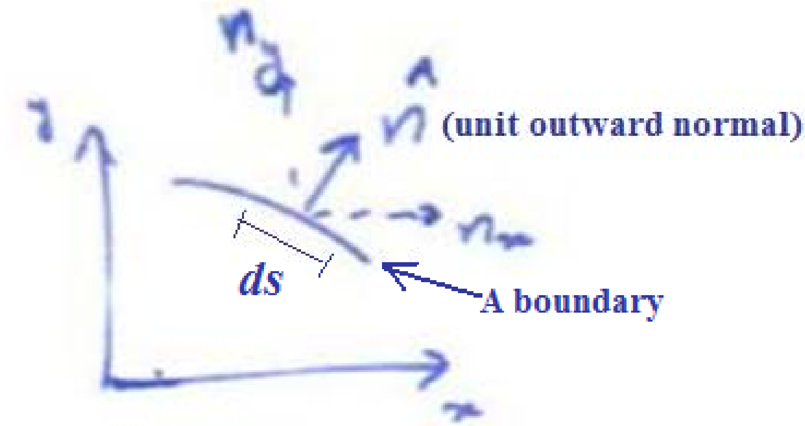
$$\therefore I(f(x,y)) = \int_{x_1}^{x_I}\int_{y_1}^{y_I} W(x,y) \times \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - f \right) dydx = 0$$

To do integration: see $W \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(W \dfrac{\partial f}{\partial x} \right) - \dfrac{\partial W}{\partial x}\dfrac{\partial f}{\partial x}$

and $W \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(W \dfrac{\partial f}{\partial y} \right) - \dfrac{\partial W}{\partial y}\dfrac{\partial f}{\partial y}$

$$\therefore I = \int\int \left[\frac{\partial}{\partial x}\left(W \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y}\left(W \frac{\partial f}{\partial y} \right) - \frac{\partial W}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} - F \right] dxdy = 0$$

Using Stokes' theorem



$$\iint \frac{\partial}{\partial x}\left(W\frac{\partial f}{\partial x}\right)dxdy = \oint W\frac{\partial f}{\partial x}n_x ds$$

$$\iint \frac{\partial}{\partial y}\left(W\frac{\partial f}{\partial y}\right)dxdy = \oint W\frac{\partial f}{\partial y}n_y ds$$

where $\hat{n} = n_x\hat{i} + n_y\hat{j}$

$$\therefore \iint\left[\frac{\partial}{\partial x}\left(W\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(W\frac{\partial f}{\partial y}\right)\right]dxdy = \oint\left(W\frac{\partial f}{\partial x}n_x + W\frac{\partial f}{\partial y}n_y\right)ds$$

Note that $\hat{n}\cdot\nabla f = (n_x\hat{i} + n_y\hat{j})\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)$

$$= n_x\frac{\partial f}{\partial x} + n_y\frac{\partial f}{\partial y} \equiv \text{flux of } f \text{ through boundary} = q_n$$

$$\therefore I = \iint \left[\frac{\partial}{\partial x}\left(W \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y}\left(W \frac{\partial f}{\partial y} \right) - \frac{\partial W}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} - F \right] dxdy = 0$$

$$= \oint_B W q_n ds - \iint \left(\frac{\partial W}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} + WF \right) dxdy = 0$$

The line integral describes the flux q_n normal to the outer boundary B of the solution domain.

$\rightarrow$ For all interior elements that do not coincide with the outer boundary, you have $\oint_B W q_n ds = 0$

$\rightarrow$ For Neumann BCs you have values for $\oint_B W q_n ds$

$\therefore$ Globally the integral will be in terms of approximate solution:

$$I(\tilde{f}(x,y)) = I^{[1,1]} + I^{[2,1]} + \cdots + I^{[i,j]} + \cdots + I^{[I-1,J-1]} + \oint_B W q_n ds = 0$$

where $I^{[i,j]} = -\iint \left(\frac{\partial W}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} + WF \right) dxdy$ for interior elements.

In Galerkin approach we use the shape functions itself as the weighing function W.

Recall, $\tilde{f}^{[i,j]}(\bar{x},\bar{y}) = f_1 \times (1 - \bar{x} - \bar{y} + \bar{x}\,\bar{y}) + f_2 \times (\bar{x} - \bar{x}\,\bar{y})$
$+ f_3 \times (\bar{x}\,\bar{y}) + f_4 \times (\bar{y} - \bar{x}\,\bar{y})$

$$\boxed{\begin{array}{l} \therefore \dfrac{\partial \tilde{f}}{\partial \bar{x}} = f_1 \times \left(-1 + \bar{y}\right) + f_2 \times \left(1 - \bar{y}\right) + f_3 \times \left(\bar{y}\right) + f_4 \times \left(-\bar{y}\right) \\ \text{Similarly, } \dfrac{\partial \tilde{f}}{\partial \bar{y}} = f_1 \times \left(-1 + \bar{x}\right) + f_2 \times \left(-\bar{x}\right) + f_3 \times \left(\bar{x}\right) + f_4 \times \left(1 - \bar{x}\right) \end{array}}$$

$\rightarrow$ As the weighing functions $N_k^{[i,j]}(\bar{x}, \bar{y})$ are applicable only in the element $\boxed{i, j}$ and elsewhere being zero we can write total integral

$$I(\tilde{f}(x,y)) = \int_{x_1}^{x_I} \int_{y_1}^{y_I} [\ \ ] dx dy = \int_{x_i}^{x_{i+1}} \int_{y_i}^{y_{i+1}} [\ \ ] dx dy$$

i.e. Integration of the $\boxed{i, j}$ element.

$\rightarrow$ Adopting local system and also the normalised coordinates for this element $\boxed{i, j}$, we need to transform

$$\int_{x_i}^{x_{i+1}} \int_{y_i}^{y_{i+1}} [\ \ ] dx dy \rightarrow \int_0^{\Delta x} \int_0^{\Delta y} [\ \ ] dx dy \rightarrow \int_0^1 \int_0^1 [\ \ ] d\bar{x} d\bar{y}$$

$\rightarrow$ In the integration the four shape functions are used

$$N_1(\bar{x},\bar{y}) = (1-\bar{x}-\bar{y}+\bar{x}\ \bar{y})$$

You can have $\dfrac{\partial N_1}{\partial \bar{x}} = -1+\bar{y};\dfrac{\partial N_1}{\partial \bar{y}} = -1+\bar{x}$

$$I = -\int_{x_i}^{x_{i+1}}\int_{y_i}^{y_{i+1}}\left[\frac{\partial W}{\partial x}\frac{\partial \tilde{f}}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial \tilde{f}}{\partial y}+WF\right]dxdy = 0$$

$$= -\int_0^1\int_0^1\left[\frac{\partial W}{\partial \bar{x}}\frac{\partial \tilde{f}}{\partial \bar{x}}\frac{1}{(\Delta x)^2}+\frac{\partial W}{\partial \bar{y}}\frac{\partial \tilde{f}}{\partial \bar{y}}\frac{1}{(\Delta y)^2}+WF\right]d\bar{x}d\bar{y}\Delta x\Delta y = 0 \rightarrow 1$$

$$\frac{\partial \tilde{f}}{\partial \bar{x}} = f_1\times(-1+\bar{y})+f_2\times(1-\bar{y})+f_3\times\bar{y}+f_4\times(-\bar{y})$$

$$\frac{\partial \tilde{f}}{\partial \bar{y}} = f_1\times(-1+\bar{x})+f_2\times(-\bar{x})+f_3\times\bar{x}+f_4\times(1-\bar{x})$$

Substitute $W = N_1$ in equation 1 and obtain first element equation

Similarly put $W = N_2$ and obtain second element equation

Again third and fourth element equations are obtained by substituting $W = N_3$ and $W = N_4$ in equation 1.

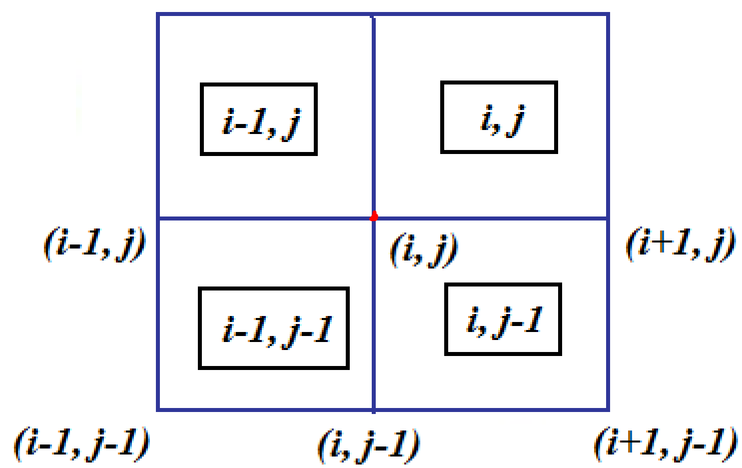
$\rightarrow$ The general node (i, j) is part of four elements

$\boxed{i-1, j-1}, \boxed{i, j-1}, \boxed{i, j}, \boxed{i-1, j}.$

$\rightarrow$ Using the above procedure, four element equations each for the remaining three elements

$\boxed{i-1, j-1}, \boxed{i, j-1},$ and $\boxed{i-1, j}$

are also generated.

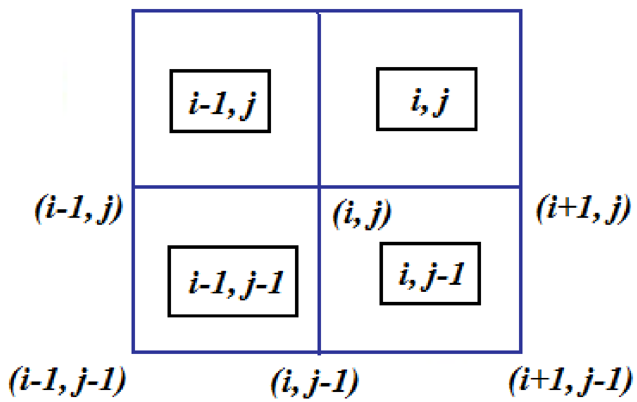


→ The corresponding element equations for which the shape functions have magnitude 1.0 at the node (i,j) are assembled to obtain nodal equations for (i, j).

→ The assembled nodal equation is subsequently applied at each of the unknown nodes in the domain.

→ System of algebraic equations in nodal values $f_{1,1}, f_{2,1}, \ldots\ldots, f_{i,j}, \ldots. f_{I,J}$ are generated.

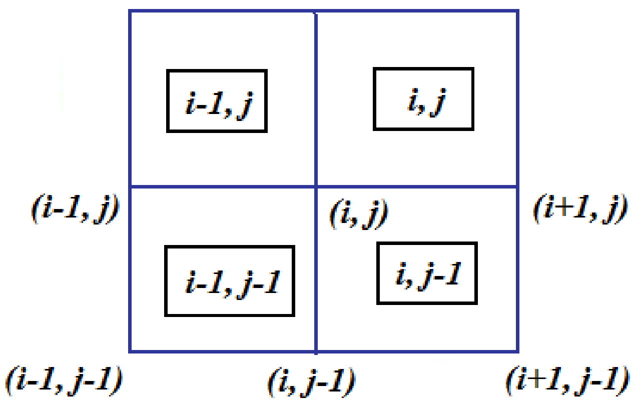
→ Adjust for boundary conditions.



$\rightarrow$ The nodal equation is modified for those elements that are part of the domain boundary by incorporating the term $\oint_B W q_n ds$

$\rightarrow$ This modification is required only for those elemnts having non-zero Neumann B.C.s

$\rightarrow$ The system of algebraic equations are solved to obtain the nodal values.



Finite-Difference Method for Hyperbolic Partial Differential equations

- As discussed, the spatial first derivative in hyperbolic PDE may be subjected to backward difference formula (Upwind scheme).
- There are Lax-Wendroff methods to solve PDEs:
- Lax and Wendroff developed $O\left(\Delta x^2\right)$ and $O\left(\Delta t^2\right)$ approximations for solving convection equations. $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$
- They suggested $\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial t^2} = u^2\frac{\partial^2 f}{\partial x^2}$

  and substituted them in Taylor's Series.

- Lax-Wendroff one step:
- Keeping base point as $f\left(x_i, t_i\right) \rightarrow f_i^{(n)}$.

$$f_i^{(n+1)} = f_i^{(n)} + \left.\frac{\partial f}{\partial t}\right|_i^{(n)} .\Delta t + \frac{1}{2!} . \left.\frac{\partial f}{\partial t}\right|_i^{(n)} .\Delta t^2 + O\left(\Delta t^3\right)$$

- Now, $\left.\frac{\partial f}{\partial t}\right|_i^{(n)} = -u \left.\frac{\partial f}{\partial x}\right|_i^{(n)}$ and $\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-u \frac{\partial f}{\partial x}\right) = -u \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = u^2 \frac{\partial^2 f}{\partial x^2}$

$$\therefore f_i^{(n+1)} = f_i^{(n)} - u \cdot \left.\frac{\partial f}{\partial x}\right|_i^{(n)} .\Delta t + \frac{1}{2} u^2 \cdot \left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n)} .\Delta t^2 + O\left(\Delta t^3\right)$$

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- Using second order centered-difference scheme:

$$f_i^{(n+1)} = f_i^{(n)} - u \cdot \frac{f_{i+1}^{(n)} - f_{i-1}^{(n)}}{2\Delta x} \cdot \Delta t + \frac{1}{2} \cdot u^2 \cdot \left(\frac{f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)}}{\Delta x^2} \right) \cdot \Delta t^2$$

- Let us define convection number as $c = \dfrac{u\Delta t}{\Delta x}$.

$$f_i^{(n+1)} = f_i^{(n)} - \frac{c}{2}\left(f_{i+1}^{(n)} - f_{i-1}^{(n)} \right) + \frac{c^2}{2} \cdot \left(f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)} \right)$$

This is the Lax-Wendroff one step approximation.

Here, $c = \dfrac{u\Delta t}{\Delta x} \leq 1.0$ for stability.