CE 601: Numerical Methods
Lecture 4

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

Gauss Elimination Method

- There are two processes in Gauss elimination.
- For a linear system *n* X *n* matrix with *n* unknowns [A]{x}={b},

→ There are (n-1) sub-steps for elimination to create the system [U]{x} = {y}

$$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - l_{ik} a_{kj}^{(k-1)} \\ b_{i}^{(k)} &= b_{i}^{(k-1)} - l_{ik} b_{k}^{(k-1)} \\ l_{ik} &= a_{ik}^{(k-1)} \big/ a_{kk}^{(k-1)} \end{aligned}\right\}$$

where $i = k+1, k+2, \ldots, n;\ j = k, k+1, k+2, \ldots, n$ and $k = 1, 2, 3, \ldots, n-1.$

- After performing (n-1) elimination steps to convert [A]{x} = {b} to form [U]{x}={y}, we need to perform back-substitution to evaluate the components of {x}.
- <u>Substitution Process</u>
- $[U]\{x\} = \{y\}$ where $\{y\}^T = \{b_1\ b_2^{(1)}\ b_3^{(2)}\ \ldots\ b_n^{(n-1)}\}$
- We start from bottom

$$x_n = b_n^{(n-1)} \Big/ a_{nn}^{(n-1)}$$

$$x_{n-1} = \left(b_{n-1}^{(n-2)} - a_{(n-1)n}^{(n-2)} x_n \right) \Big/ a_{(n-1)(n-1)}^{(n-2)}$$

$$x_{n-2} = \left(b_{n-2}^{(n-3)} - a_{(n-2)(n-1)}^{(n-3)} x_{n-1} - a_{(n-2)n}^{(n-3)} x_n \right) \Big/ a_{(n-2)(n-2)}^{(n-3)}$$

- Similarly you get

$$x_i = \frac{b_i^{(i-1)} - a_{i(i+1)}^{(i-1)} x_{i+1} - a_{i(i+2)}^{(i-1)} x_{i+2} - \cdots - a_{in}^{(i-1)} x_n}{a_{ii}^{(i-1)}}$$

$$\text{i.e.,}\, x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}};\; i = (n-1), (n-2), \ldots, 2, 1.$$

- <u>Operations Involved in Gauss Elimination</u>
- While using computational methods to solve linear systems, emphasis should be given on efficient way of computing using the algorithms.
- The normal computing operations involved are
    - ❖ Addition
    - ❖ Subtraction
    - ❖ Multiplication
    - ❖ Division
- Let us hypothetically suggest for a computer, each of the above operation involves say ‘t’ mili-seconds. So if we have large number of above operattions, the computer will take more time to evaluate.
- In the case of Gauss elimination method, let us see how many no. of operations are involved to solve an n X n linear system.

- No. of operations involved in elimination steps.

→ No. of elimination steps = (n-1)

→ No. of operations for first elimination step (i.e. k=1)

→ Evaluate multiplication factor l_{i1}.

→ There are (n-1) rows to be operated in first sub-step, so no of operations for multiplication factors,

$$l_{21} = a_{21}/a_{11}, l_{31} = a_{31}/a_{11}, \ldots, l_{i1} = a_{i1}/a_{11}, \ldots l_{n1} = a_{n1}/a_{11}$$

i.e. (n-1) operations (all divisions)

→ For evaluating $a_{ij}^{(1)} = a_{ij} - l_{i1}a_{1j}$

(2 operations)*(n-1)*(n-1)

→ For evaluating $b_i^{(1)} = b_i - l_{i1}b_1$

(2 operations)*(n-1)

- In the first elimination step, you have the following number of operations.

$$= (n-1) + 2(n-1)^2 + 2(n-1)$$

$$= (n-1)(2n+1)$$

- Similarly, in the second elimination step (k=2)

→The no. of operations = (n-1)(2n+1),

→For k=3, no. of operations = (n-3)(2n-3)

- In general for any k^{th} elimination step, we have no. of operations = (n-k)(2n-2k+3)
- Total no. operations for elimination

$$= \sum_{p=1}^{n-1} (n-p)(2n-2p+3)$$

$$= \frac{1}{6}(n-1)(7+4n)n$$

$$= \frac{2}{3}n^3 + \frac{n^2}{2} - \frac{7}{6}n$$

- No. of operations involved in back-substitution. There are 'n' back-substiution steps: i.e. *i=1,2,3,…,n*

→ First back sub step,

$x_n = b_n^{(n-1)} / a_{nn}^{(n-1)}$ -> 1 operation

→ Second back sub step,

$x_{n-1} = \left(b_{n-1}^{(n-2)} - a_{(n-1)n}^{(n-2)} x_n \right) / a_{(n-1)(n-1)}^{(n-2)}$ -> 3 operations

→ In general for any 'i',

$$x_{n-2} = \left(b_{n-2}^{(n-3)} - a_{(n-2)(n-1)}^{(n-3)} x_{n-1} - a_{(n-2)n}^{(n-3)} x_n \right) / a_{(n-2)(n-2)}^{(n-3)}$$

$i = n-1, n-2, \ldots, 1$ -> (2i-1) operations.

- Total no. of operations for back-substitution: $\sum_{i=1}^{n} (2i-1) = n^2$
- Total no. of operations in entire Gauss elimination process, $= \frac{2}{3}n^3 + \frac{n^2}{2} - \frac{7}{6}n + n^2$

$$= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

- E.g. If you have 1000 X 1000 linear system, total no. of operations = 0.6681655×10^{-5}.

  If an operation takes (hypothetically) 0.1 milli seconds per operation, then total time taken = 66817 seconds ≈ 18.6 hours.
- That is why, we suggest to have efficient computer methods.
- Gauss elimination method is a traditional form, however, it is not the efficient method to solve system of linear equation.
- There is another direct elimination method called Gauss-Jordan elimination method. (this I request you to refer on your own).
- In Gauss-Jordan method the principle is the convert [A]{x} = {b} to the form [I]{x}={x} where {I} is a identity matrix.
- Gauss-Jordan method is computationally not efficient. You will see that the number of operations involved in Gauss-Jordan is $= n^3 + n^2 - n$.

- In school days you have also studied matrix inverse methods and corresponding determinants to solve linear systems. This method take $2n^3 - 2n^2 + n$ number of arithmetic operations for matrix inverse. Also the multiplication $[A]^{-1}\{b\}$ further requires $2n^2-n$ operations.
- Note: As discussed in one of the earlier lecture, numerical methods may generate errors in the solutions.
- The gauss elimination method may also be present with errors in the solutions like round-off errors. One can use partial pivoting or scaled partial pivoting to reduce such errors.

- LU Decomposition

- We have discussed that matrix can be factored i.e., it can be given as product of two different matrix.

  [A] = [B][C]

- There can be many possibilities of obtaining factor matrices.

- In a similar tone, one can also factorize [A] as product of [L] and [U] i.e., [A]= [L][U] where [L] is lower triangular and [U] is a upper triangular matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \ldots .(4)$$

- In the representation (Eq.4) if we specify the values of diagonal elements of either [L] or [U], then that factorization will be unique.
- The LU decomposition methods the Doolittle and Crout’s work on these principles