CE 601: Numerical Methods

Lecture 39

Galerkin FEM for Laplace Equation-1

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Galerkin FEM

Here, we will see how Galerkin FEM can be applied for 2-D cases.

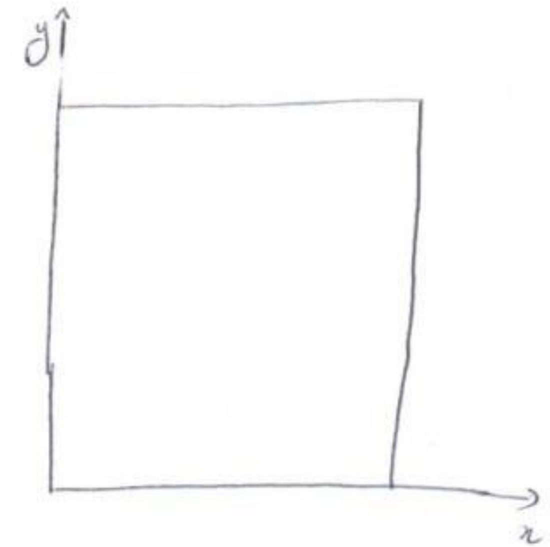
→ We will see Galerkin FEM to solve 2-D Laplace equation (or Poisson equation).

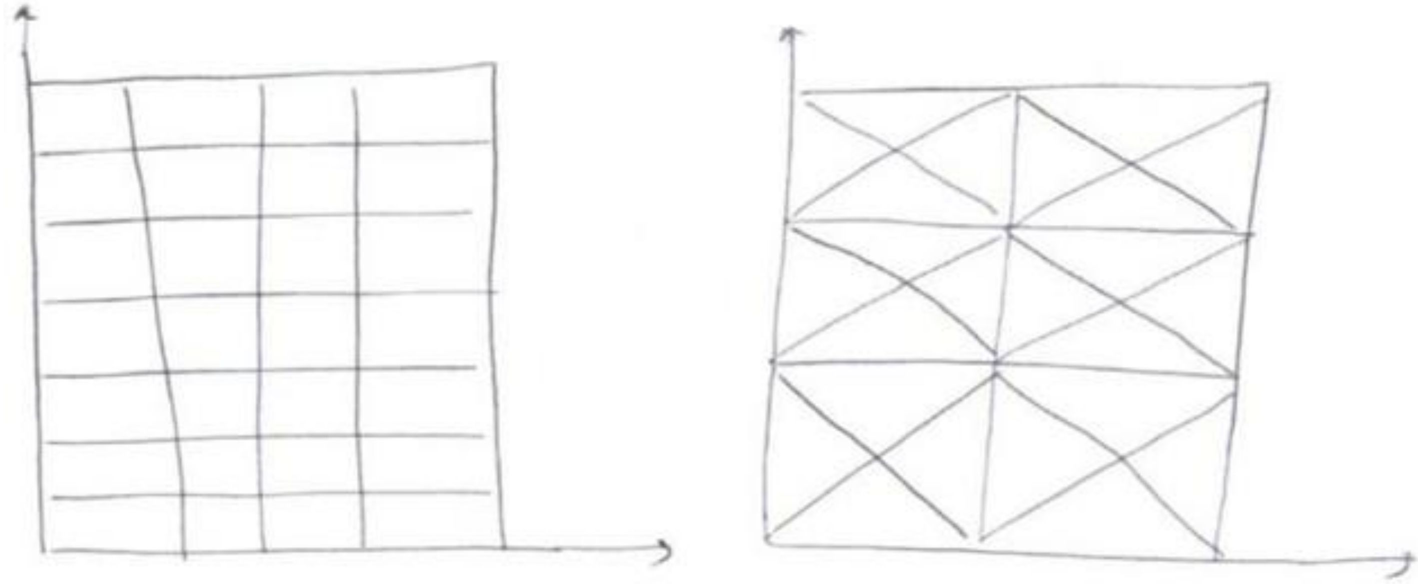
$$\left.\begin{aligned} &\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=F(x,y) \\ \text{i.e.}\qquad &\qquad\text{or} \\ &\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0 \end{aligned}\right\}\text{ with appropriate BCs.}$$

→ Let the two dimensional domain be rectangular in shape.

→ Figure shows the domain.

→ First thing is we need to discretise the domain into small small elements.





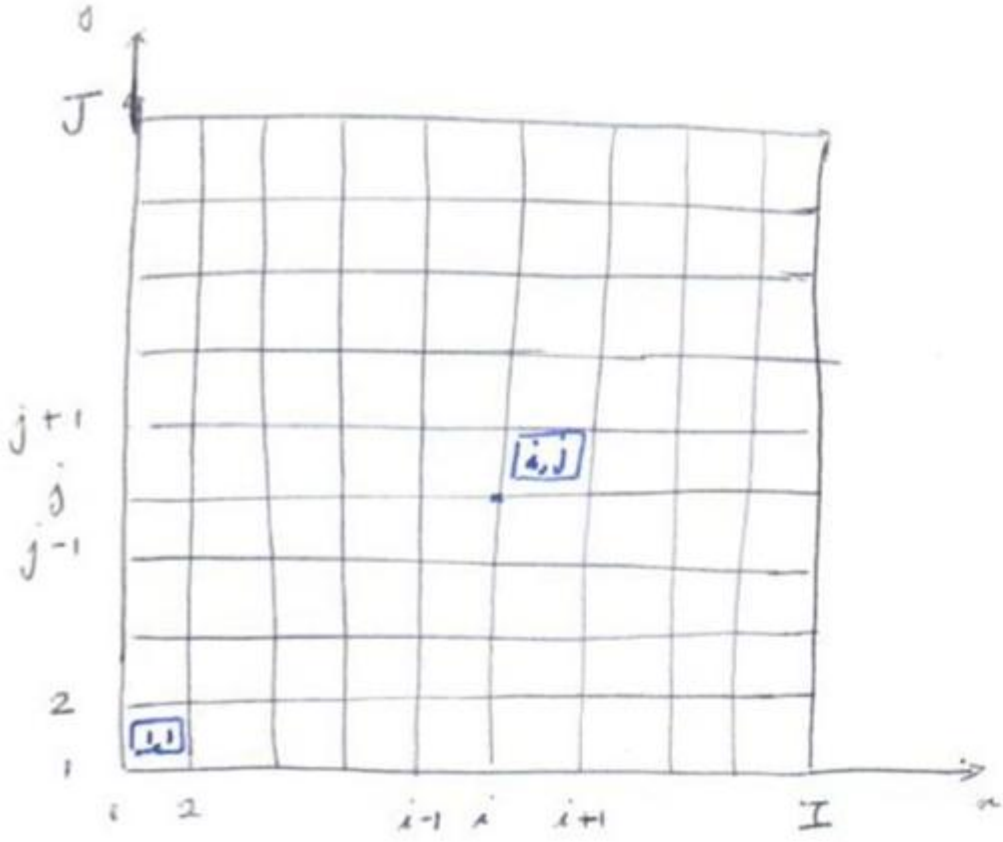
→ The domain discretised using "rectangular elements", "triangular elements", "quadrilateral elements", etc.

→ You have the freedom to choose any shape for the element.

→ In our case let us consider that the domain is discretised using rectangular elements.

∴ The $x-$axis is discretised $i=1,2,3,...,I$ nodes or discrete points

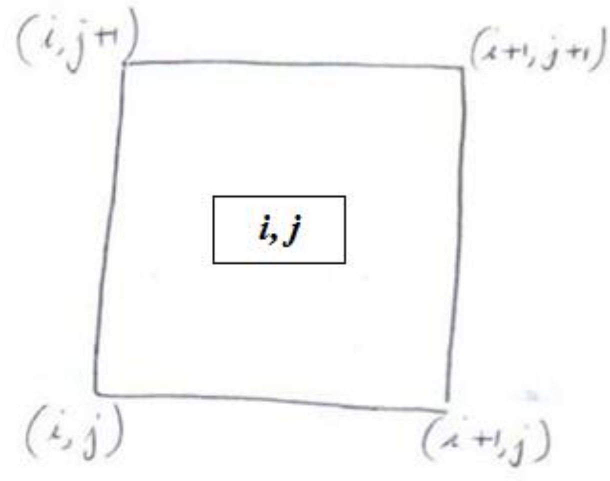
The $y-$axis is discretised $j=1,2,3,...,J$ nodes.



$\therefore$ For this rectangular domain there are a total of $(I \times J)$ discrete nodes.

$\rightarrow$ Any general node is given as (i, j) in the suffix.

$\rightarrow$ Four nodes constitute an rectangular element. There are a total of $(I-1)\times(J-1)$ elements. Any general element is given as $\boxed{i, j}$ in the superfix.



$\rightarrow$ As described for the one-dimensional case, there will be combination of approximating polynomial for $'f'$ in the entire domain

i.e. $f(x, y) \approx \tilde{f}(x, y) = \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} \tilde{f}^{[i,j]}(x, y)$

(i.e. Sum of series of local interpolating polynomials)

$\rightarrow$ For an element $\boxed{i, j} \rightarrow$ the approximation is $\tilde{f}^{[i,j]}(x, y)$

$\rightarrow$ As there are four nodes $(i, j), (i+1, j), (i+1, j+1)$ and $(i, j+1)$ associated with the element $\boxed{i, j}$, the approximation is given as:

$$\tilde{f}^{[i,j]}(x, y) = f_1 N_1(x, y) + f_2 N_2(x, y) + f_3 N_3(x, y) + f_4 N_4(x, y)$$

i.e. for the element $\boxed{i, j}$, the global node numbers are replaced by local node numbers $1, 2, 3, 4$ and N_1, N_2, N_3, N_4 are shape functions for the element $\boxed{i, j}$.

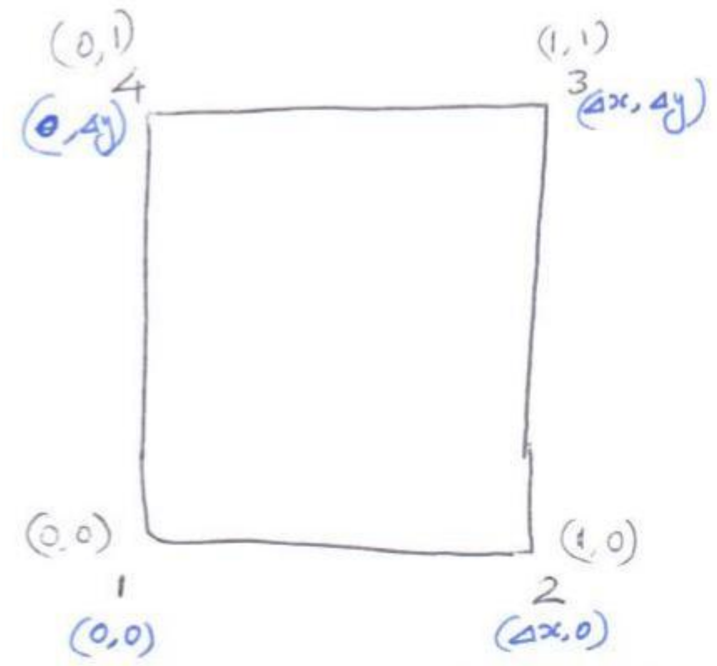
The shape functions are:

$$N_1(x,y) = \begin{cases} 1.0 \text{ at node } 1 \\ 0 \text{ at other nodes} \end{cases}, N_2(x,y) = \begin{cases} 1.0 \text{ at node } 2 \\ 0 \text{ at other nodes} \end{cases}$$

$$N_3(x,y) = \begin{cases} 1.0 \text{ at node } 3 \\ 0 \text{ at other nodes} \end{cases}, N_4(x,y) = \begin{cases} 1.0 \text{ at node } 4 \\ 0 \text{ at other nodes} \end{cases}$$

$\rightarrow$ Let us assume the shape functions are linear

i.e. $N(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$



$\rightarrow$ As the shape functions for the element $\boxed{i,j}$ have to be fomalised, we use the normalised values of x and y within the element.

i.e. $\bar{x} = \dfrac{x}{\Delta x}$ and $\bar{y} = \dfrac{y}{\Delta y}$

$\rightarrow$ The local coordinates at the node $(i, j) \rightarrow (x, y) = (0, 0)$, $(i+1, j) \rightarrow (\Delta x, 0), (i+1, j+1) \rightarrow (\Delta x, \Delta y)$, and $(i, j+1) \rightarrow (0, \Delta y)$.

$\rightarrow$ The normalised coordinates are therefore for the local element

$(0,0) \rightarrow (0,0)$, $(\Delta x, 0) \rightarrow (1, 0)$

$(\Delta x, \Delta y) \rightarrow (1,1)$ and $(0, \Delta y) \rightarrow (0,1)$

We develop shape functions using normalised coordinates $(\bar{x}, \bar{y})$.

$$N_1(\bar{x}, \bar{y}) = a_0 + a_1\bar{x} + a_2\bar{y} + a_3\bar{x}\,\bar{y}$$

$$N_1(0, 0) = 1.0 = a_0 + 0 + 0 + 0; \therefore a_0 = 1.0$$

$$N_1(1, 0) = 0.0 = a_0 + a_1 + 0 + 0; \therefore a_1 = -1.0$$

$$N_1(0,1) = 0.0 = a_0 + 0 + a_2 + 0; \therefore a_2 = -1.0$$

$$N_1(1,1) = 0.0 = a_0 + a_1 + a_2 + a_3; \therefore a_3 = 1.0$$

$$\therefore N_1(\bar{x}, \bar{y}) = 1.0 - \bar{x} - \bar{y} + \bar{x}\,\bar{y}$$

Similarly, $N_2(\bar{x},\bar{y}) = a_0 + a_1\bar{x} + a_2\bar{y} + a_3\bar{x}\,\bar{y}$

We have $N_2(0,0) = 0.0 = a_0; \therefore a_0 = 0.0$

$$N_2(1,0) = 1.0 = a_0 + a_1; \therefore a_1 = 1.0$$

$$N_2(0,1) = 0.0 = a_0 + a_2; \therefore a_2 = 0.0$$

$$N_2(1,1) = 0.0 = a_0 + a_1 + a_2 + a_3; \therefore a_3 = -1.0$$

i.e. $N_2(\bar{x},\bar{y}) = \bar{x} - \bar{x}\,\bar{y}$

Similarly, $N_3(\bar{x},\bar{y}) = \bar{x}\,\bar{y}$

and $N_4(\bar{x},\bar{y}) = \bar{y} - \bar{x}\,\bar{y}$

$$\therefore \tilde{f}^{[i,j]}(x,y) = f_1 \times (1 - \bar{x} - \bar{y} + \bar{x}\,\bar{y}) + f_2 \times (\bar{x} - \bar{x}\,\bar{y}) + f_3 \times (\bar{x}\,\bar{y}) + f_4 \times (\bar{y} - \bar{x}\,\bar{y})$$

This is the interpolating polynomial for the element $\boxed{i, j}$.

→ Define the Residual for the problem and evaluate weighted integral.

$$R(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - F$$

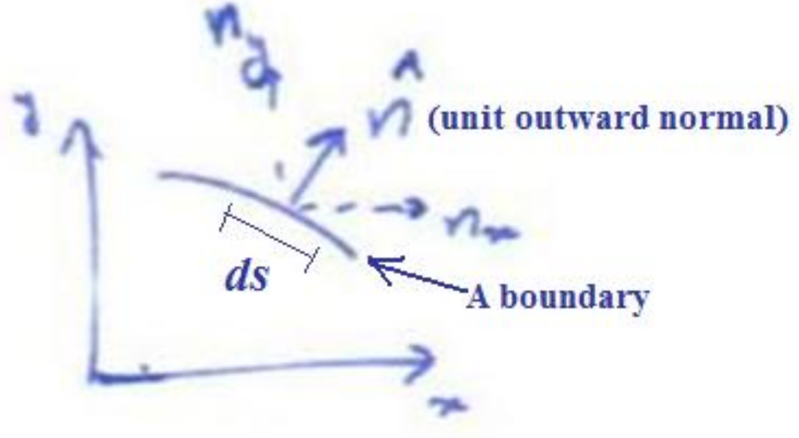
$$\therefore I(f(x,y)) = \int_{x_1}^{x_I}\int_{y_1}^{y_I} W(x,y) \times \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - f \right) dy dx = 0$$

To do integration: see $W\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(W \dfrac{\partial f}{\partial x} \right) - \dfrac{\partial W}{\partial x}\dfrac{\partial f}{\partial x}$

and $W\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(W \dfrac{\partial f}{\partial y} \right) - \dfrac{\partial W}{\partial y}\dfrac{\partial f}{\partial y}$

$$\therefore I = \int\int \left[\frac{\partial}{\partial x}\left(W \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y}\left(W \frac{\partial f}{\partial y} \right) - \frac{\partial W}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} - F \right] dx dy = 0$$

Using Stokes' theorem



$$\iint \frac{\partial}{\partial x}\left(W\frac{\partial f}{\partial x}\right) dxdy = \oint W\frac{\partial f}{\partial x} n_x ds$$

$$\iint \frac{\partial}{\partial y}\left(W\frac{\partial f}{\partial y}\right) dxdy = \oint W\frac{\partial f}{\partial y} n_y ds$$

where $\hat{n} = n_x \hat{i} + n_y \hat{j}$

$$\therefore \iint \left[\frac{\partial}{\partial x}\left(W\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial f}{\partial y}\right)\right] dxdy = \oint \left(W\frac{\partial f}{\partial x} n_x + W\frac{\partial f}{\partial y} n_y \right) ds$$

Note that $\hat{n}\cdot\nabla f = (n_x \hat{i} + n_y \hat{j})\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)$

$$= n_x \frac{\partial f}{\partial x} + n_y \frac{\partial f}{\partial y} \equiv \text{flux of } f \text{ through boundary} = q_n$$

$$\therefore I = \iint \left[\frac{\partial}{\partial x}\left(W \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y}\left(W \frac{\partial f}{\partial y} \right) - \frac{\partial W}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} - F \right] dxdy = 0$$

$$= \oint_B W q_n ds - \iint \left(\frac{\partial W}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial f}{\partial y} + WF \right) dxdy = 0$$

The line integral describes the flux q_n normal to the outer boundary B of the solution domain.

$\rightarrow$ For all interior elements that do not coincide with the outer boundary, you have $\oint_B W q_n ds = 0$

$\rightarrow$ For Neumann BCs you have values for $\oint_B W q_n ds$