CE 601: Numerical Methods

Lecture 38

Galerkin FEM

Example

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Example: Solve using Galerkin FEM the boundary-value ODE

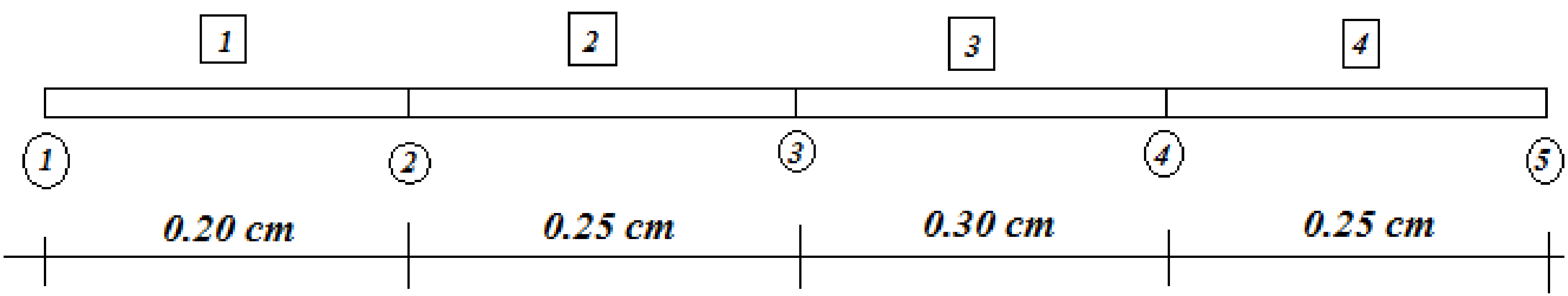
$$\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_a; \; T(0.0) = 100^\circ C, \; \left.\frac{dT}{dx}\right|_{(1.0)} = 0.0, \; 0 \le x \le 1.0$$

where $-\alpha^2 = -16.0 \text{ cm}^{-2}, T_a = 0.0^\circ C.$

Solution.

→ To solve this BV-ODE, we will be using the Galerkin FEM.

→ The domain is non-uniformly discretised as such:



$\rightarrow$ There are 5-nodes and 4-elements.

$\rightarrow$ The differential equation is:

$$\frac{d^2T}{dx^2} - 16T = 0.0; \; T(0) = 100^o C, \left.\frac{dT}{dx}\right|_{(5)} = 0.0$$

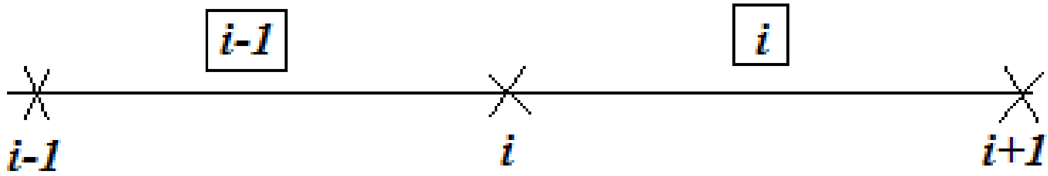
$\therefore$ We will apply the elemental equations for each elements and then assemble to form the nodal equation for the unknown node.

$\Rightarrow$ For element $\boxed{i}$ the element equations are:

$$-T_i\left[\frac{1}{\Delta x^{(i)}} - \frac{\overline{Q}^{(i)}\Delta x^{(i)}}{3}\right] + T_{i+1}\left[\frac{1}{\Delta x^{(i)}} + \frac{\overline{Q}^{(i)}\Delta x^{(i)}}{6}\right] - \frac{\overline{F}^{(i)}\Delta x^{(i)}}{2} = 0 \;\rightarrow (1A)$$

$$T_i\left[\frac{1}{\Delta x^{(i)}} + \frac{\overline{Q}^{(i)}\Delta x^{(i)}}{6}\right] - T_{i+1}\left[\frac{1}{\Delta x^{(i)}} - \frac{\overline{Q}^{(i)}\Delta x^{(i)}}{3}\right] - \frac{\overline{F}^{(i)}\Delta x^{(i)}}{2} = 0 \;\rightarrow (1B)$$

$\Rightarrow$ To assemble element equations for a node 'i'.



Node 'i' is part of element $\boxed{i-1}$ and element $\boxed{i}$.

For element $\boxed{i-1}$, the element equations are:

$$-T_{i-1}\left[\frac{1}{\Delta x^{(i-1)}}-\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{3}\right]+T_i\left[\frac{1}{\Delta x^{(i-1)}}+\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{6}\right]-\frac{\overline{F}^{(i-1)}\Delta x^{(i-1)}}{2}=0 \rightarrow \text{(2A)}$$

$$T_{i-1}\left[\frac{1}{\Delta x^{(i-1)}}+\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{6}\right]-T_i\left[\frac{1}{\Delta x^{(i-1)}}-\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{3}\right]-\frac{\overline{F}^{(i-1)}\Delta x^{(i-1)}}{2}=0 \quad \rightarrow \text{(2B)}$$

$\Rightarrow$ Eqn.(2B) represents element equation with shape function $N_i^{(i-1)}(x=x_i)=1.0$

Similarly (1A) represents element equation for element $\boxed{i}$ with shape function

$N_i^{(i)}(x=x_i)=1.0$

We need to add (1A) and (2B) to assemble and form nodal equation for node 'i'.

$\Rightarrow$ On assembling the nodal equation for node $'i'$ will be

$$T_{i-1}\left[\frac{1}{\Delta x^{(i-1)}}+\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{6}\right]-T_i\left[\frac{1}{\Delta x^{(i-1)}}-\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{3}\right]-\frac{\overline{F}^{(i-1)}\Delta x^{(i-1)}}{2}$$

$$-T_i\left[\frac{1}{\Delta x^{(i)}}-\frac{\overline{Q}^{(i)}\Delta x^{(i)}}{3}\right]+T_{i+1}\left[\frac{1}{\Delta x^{(i)}}+\frac{\overline{Q}^{(i)}\Delta x^{(i)}}{6}\right]-\frac{\overline{F}^{(i)}\Delta x^{(i)}}{2}=0$$

i.e.

$$\boxed{\begin{aligned}&T_{i-1}\left[\frac{1}{\Delta x^{(i-1)}}+\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{6}\right]-T_i\left[\frac{1}{\Delta x^{(i-1)}}+\frac{1}{\Delta x^{(i)}}-\frac{\overline{Q}^{(i-1)}\Delta x^{(i-1)}}{3}-\frac{\overline{Q}^{(i)}\Delta x^{(i)}}{3}\right]\\&+T_{i+1}\left[\frac{1}{\Delta x^{(i)}}+\frac{\overline{Q}^{(i)}\Delta x^{(i)}}{6}\right]=\frac{1}{2}\left[\overline{F}^{(i-1)}\Delta x^{(i-1)}+\overline{F}^{(i)}\Delta x^{(i)}\right]\end{aligned}}\rightarrow(3)$$

$\Rightarrow$ We need to apply this node equation in each of the unknown nodes say here T_2, T_3, T_4 and T_5. Already $T_1 = 100^o C$.

For node (2): $T_1\left[\frac{1}{0.2}+\frac{-16}{6}\times 0.2\right]-T_2\left[\frac{1}{0.2}+\frac{1}{0.25}-\frac{-16}{3}\times 0.2-\frac{-16}{3}\times 0.25\right]$

$+T_3\left[\frac{1}{0.25}+\frac{-16}{6}\times 0.25\right]=\frac{1}{2}\times 0.0=0.0$

$\Rightarrow$ Note that weighing (or shape function) $N_2^{(1)}(x=x_1=0)=0.0$

$\therefore W_j y_a'=0.0$ for element $\boxed{1}$.

i.e. $-T_2\times 11.4+T_3\times 3.3333=-446.667 \qquad \rightarrow (4A)$

For node (3): $T_2\left[\frac{1}{0.25}+\frac{-16}{6}\times 0.25\right]-T_3\left[\frac{1}{0.25}+\frac{1}{0.30}-\frac{-16}{3}\times 0.25-\frac{-16}{3}\times 0.30\right]$

$+T_4\left[\frac{1}{0.30}+\frac{-16}{6}\times 0.30\right]=0.0$

i.e. $T_2\times 3.333-T_3\times 10.2667+T_4\times 2.5333=0.0 \quad \rightarrow (4B)$

For node (4): $T_3\left[\frac{1}{0.30}+\frac{-16}{6}\times 0.30\right]-T_4\left[\frac{1}{0.30}+\frac{1}{0.25}-\frac{-16}{3}\times 0.30-\frac{-16}{3}\times 0.25\right]$

$+T_5\left[\frac{1}{0.25}+\frac{-16}{6}\times 0.25\right]=0.0$

i.e. $T_3 \times 2.5333 - T_4 \times 9.7333 + T_5 \times 3.3333 = 0.0 \quad \rightarrow (4C)$

For node (5): We need to use the Neumann B.C. here.

The element equation for element $\boxed{4}$ with shape function

$N_5^{(4)}(x = x_5 = 1.0) = 1.0$ is:

$$T_{I-1}\left[\frac{1}{\Delta x^{(I-1)}} + \frac{\overline{Q}^{(I-1)}}{6}(\Delta x^{(I-1)})\right] - T_I\left[\frac{1}{\Delta x^{(I-1)}} - \frac{\overline{Q}^{(I-1)}}{3}(\Delta x^{(I-1)})\right]$$

$$-\overline{F}^{(I-1)}\frac{\Delta x^{(I-1)}}{2} - W_I\, T'\big|_b = 0.0$$

Since $T'\big|_b = 0.0$ i.e. $\left.\dfrac{dT}{dx}\right|_{x=1.0} = 0.0$, the last term vanishes.

$$\therefore T_4\left[\frac{1}{0.25} + \frac{-16}{6}\times 0.25\right] - T_5\left[\frac{1}{0.25} - \frac{-16.0}{3}\times 0.25\right] = 0$$

i.e. $3.3333T_4 - 5.3333T_5 = 0.0$

So you have four equations and 4 unknowns. Solve.

$$\begin{bmatrix} -11.4 & 3.3333 & 0 & 0 \\ 3.3333 & -10.2667 & 2.5333 & 0 \\ 0 & 2.5333 & -9.7333 & 3.3333 \\ 0 & 0 & 3.3333 & -5.3333 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} -446.667 \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

Use any appropriate method to solve this system of linear equations to get the nodal values T_i.