CE 601: Numerical Methods

Lecture 37

Galerkin FEM

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In the last class, we started the discussion on Galerkin FEM.

We were considering the boundary-value ODE

$$\frac{d^2 y}{dx^2} + Qy = F; \text{ with appropriate BCs}$$

In a nutshell, the procedure is

i) Discretise the global solution domain $D(x)$ into elements. There can be $I-$nodes and $I-1$ elements.

ii) Approximate $y(x) \approx \tilde{y}(x) =$ combinations of interpolating functions.

iii) Substitute $\tilde{y}(x)$ in the expression for residual i.e. $R(x) = \frac{d^2 \tilde{y}}{dx^2} + Q\tilde{y} - F$

iv) The weighted integral is to be evaluated $I = \int W(x)R(x)dx$

v) Determine element equations for each element.

vi) Assemble element equations to form system equations.

vii) Adjust the system equations by considering boundary conditions.

viii) Solve the adjusted system equation for the nodal values y_i.

1) <u>The Domain Discretisation</u>

The domain $D(x)$ discretised: $I \rightarrow$ nodes and $I-1 \rightarrow$ elements.

The global solution of $\dfrac{d^2 y}{dx^2} + Qy = F$ is $y(x)$.

Let us approximate this global solution

$$y(x) \approx \tilde{y}(x) = \tilde{y}^{(1)}(x) + \tilde{y}^{(2)}(x) + \cdots + \tilde{y}^{(i)}(x) + \cdots + \tilde{y}^{(I-1)}(x)$$

This means that $\tilde{y}(x)$ is sum of series of local interpolating polynomials $\tilde{y}^{(i)}(x); i = 1, 2, 3, \ldots, I-1$. These local interpolating polynomials are valid only within the element $'i'$, elsewhere it is zero.

2) We write $\tilde{y}^{(i)}(x) = y_i N_i^{(i)}(x) + y_i N_{i+1}^{i)}(x)$

$$\left.\begin{aligned} \text{Recall } N_i^{(i)}(x) &= -\frac{x - x_{i+1}}{\Delta x^{(i)}} \\ N_{i+1}^{(i)}(x) &= -\frac{x - x_i}{\Delta x^{(i)}} \end{aligned}\right\} \text{Linear Shape functions}$$

You can also use higher order shape functions as well.

You can see that the shape function $N_i^{(i)}(x)$ is valid only for element $'i'$ and elsewhere it is zero.

3) To use Galerkin method

$$I(\tilde{y}(x)) = \int_a^b W_j(x)R(x)dx = \int_a^b W_j(x)\left[\tilde{y}''+Q\tilde{y}-F\right]dx = 0$$

Now we have seen in the last class that

$$I(\tilde{y}(x)) = \int_a^b \left[-\tilde{y}'W_j'+Q\tilde{y}W_j-FW_j\right]dx+\tilde{y}_b'W_j(b)-\tilde{y}_a'W_j(a)=0$$

i.e. $$I(\tilde{y}(x)) = I^{(1)}(\tilde{y}(x))+I^{(2)}(\tilde{y}(x))+\cdots+I^{(I-1)}(\tilde{y}(x))$$
$$+\tilde{y}_b'W_I(b)-\tilde{y}_a'W_I(a)=0$$

where $$I^{(i)}(\tilde{y}(x)) = \int_{x_i}^{x_{i+1}} \left[-\tilde{y}'W_j'+Q\tilde{y}W_j-FW_j\right]dx$$

$\tilde{y}^{(i)}(x)$ is the interpolating polynomial in the element i.

$$\tilde{y}^{(i)}(x) = y_i N_i^{(i)}(x) + y_{i+1} N_{i+1}^{(i)}(x) = -\frac{x - x_{i+1}}{\Delta x^{(i)}} y_i + -\frac{x - x_i}{\Delta x^{(i)}} y_{i+1}$$

As suggested earlier Galerkin recommended use of shape functions as weighing functions.

As $N_i^{(i)}(x) = 0.0$ for $x > x_{i+1}$ & $x < x_i$

$N_{i+1}^{(i)}(x) = 0.0$ for $x > x_{i+1}$ & $x < x_i$

We can say that the total integral using $W_j = N_i^{(i)}$

$$I(\tilde{y}(x)) = \int_a^b (\quad) dx = \int_{x_i}^{x_{i+1}} \left[-\tilde{y}' \frac{d}{dx} (N_i^{(i)}(x)) + Q\tilde{y} N_i^{(i)} - F N_i^{(i)} \right] dx = 0$$

$\because N_i^{(i)}(x) = 0$ everywhere else other than $x_i < x < x_{i+1}$

Also $N_i^{(i)}(a) = 0$ and $N_i^{(i)}(b) = 0$

Similarly for $W_j = N_{i+1}^{(i)}(x)$

$$I(\tilde{y}(x)) = \int_{x_i}^{x_{i+1}} \left[-\tilde{y}' \frac{d}{dx} (N_{i+1}^{(i)}(x)) + Q\tilde{y} N_{i+1}^{(i)} - F N_{i+1}^{(i)} \right] dx = 0$$

We now get two element equations for the element $'i'$.

Now since, $N_i^{(i)}(x) = -\dfrac{x - x_{i+1}}{\Delta x^{(i)}} \therefore \dfrac{dN_i^{(i)}}{dx} = -\dfrac{1}{\Delta x^{(i)}}$

Smilarly, $\dfrac{dN_{i+1}^{(i)}}{dx} = \dfrac{1}{\Delta x^{(i)}}$.

$\therefore$ In the two element equations:

$$I(\tilde{y}(x)) = \int_{x_i}^{x_{i+1}} \left[-\tilde{y}' \frac{d}{dx}(N_i^{(i)}(x)) + Q\tilde{y}N_i^{(i)} - FN_i^{(i)} \right] dx = 0$$

$$= \int_{x_i}^{x_{i+1}} \left[-\tilde{y}' \times \left(-\frac{1}{\Delta x^{(i)}} \right) + Q\tilde{y} \times \left(-\frac{x - x_{i+1}}{\Delta x^{(i)}} \right) - F \times \left(-\frac{x - x_{i+1}}{\Delta x^{(i)}} \right) \right] dx = 0$$

i.e. $$\frac{1}{\Delta x^{(i)}} \left[\int_{x_i}^{x_{i+1}} \tilde{y}' dx - \int_{x_i}^{x_{i+1}} Q\tilde{y}(x - x_{i+1}) dx + \int_{x_i}^{x_{i+1}} F(x - x_{i+1}) dx \right] = 0$$

Similarly, $I(\tilde{y}(x)) = \int_{x_i}^{x_{i+1}} \left[-\tilde{y}' \times \frac{1}{\Delta x^{(i)}} + Q\tilde{y} \times \frac{(x - x_i)}{\Delta x^{(i)}} - F \times \frac{(x - x_i)}{\Delta x^{(i)}} \right] dx = 0$

or, $\frac{1}{\Delta x^{(i)}} \left[\int_{x_i}^{x_{i+1}} -\tilde{y}' dx + \int_{x_i}^{x_{i+1}} Q\tilde{y}(x - x_i) dx - \int_{x_i}^{x_{i+1}} F(x - x_i) dx \right] = 0$

Using $Q(x)$ and $F(x)$ as average values for each element

i.e. $\bar{Q}^{(i)}(x)$ and $\bar{F}^{(i)}(x)$, we get

$$\boxed{\begin{aligned} & -y_i \left[\frac{1}{\Delta x^{(i)}} - \bar{Q}^{(i)} \frac{\Delta x^{(i)}}{3} \right] + y_{i+1} \left[\frac{1}{\Delta x^{(i)}} + \bar{Q}^{(i)} \frac{\Delta x^{(i)}}{6} \right] - \bar{F}^{(i)} \frac{\Delta x^{(i)}}{2} = 0 \\ & y_i \left[\frac{1}{\Delta x^{(i)}} + \bar{Q}^{(i)} \frac{\Delta x^{(i)}}{6} \right] - y_{i+1} \left[\frac{1}{\Delta x^{(i)}} - \bar{Q}^{(i)} \frac{\Delta x^{(i)}}{3} \right] - \bar{F}^{(i)} \frac{\Delta x^{(i)}}{2} = 0 \end{aligned}}$$

These are the two element equations in algebraic form.

Assemble these element equations for all the elements starting from element $\boxed{1}$.