CE 601: Numerical Methods

Lecture 36

Finite Element Methods

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In the last class, we discussed on the approximate methods to solve a boundary-value problem (say)

$$\frac{d^2y}{dx^2} + Qy = F; \text{ Appropriate BCs}$$

They were

* Rayleigh-Ritz method based on calculus of variations.
* Collocation method based on residuals.
* Galerkin weighted residual method.

In Galerkin weighted residual method, the methodology involves:

- Identify the differential equation to be solved (e.g. here $\frac{d^2y}{dx^2} + Qy = F$)
- Approximate the actual solution $y(x)$ with apprximate soln.

$\tilde{y}(x)$, where $\tilde{y}(x) = \sum_{i=1}^{I} C_i y_i$

y_i are trial functions usually linearly independent polynomials that satisfy the given boundary conditions.

- Define the residual functions $W_j(x)$; $j = 1, 2, 3, \ldots$

galerkin suggested that results will be better if we choose the trial functions $y_i(x)$ as the weighing functions.

- Perform the integration using each weighing function

i.e. $\int_{x_1}^{x_2} W_j(x) R(x) dx = 0$; $j = 1, 2, 3, \ldots$

- Solve the system of weighted residual integrals for the coefficients

C_i; $(i = 1, 2, 3, \ldots)$

$\Rightarrow$ Usually collocation method is not that much used.

$*$ If the variational functional is known a priori and if we want to solve in the domain (usually solid mechanics problem), Rayleigh-Ritz method preferred.

$*$ If the differential equation is already specified in the domain, the Galerkin weighted residual method is preferred (fluid mechanics).

The Finite-Element Method for B.V. problems

$\rightarrow$ The Rayleigh-Ritz method and Galerkin weighted residual method approximate the solution $\tilde{y}(x)$ for the entire domain.

$\rightarrow$ As linearly independent trial functions are applied for the whole domain $D(x),$ the accuracy falters for larger domains.

$\rightarrow$ Or else you might have to use higher degree polynomials as trial functions.

$\rightarrow$ It is a natural approach to increase the degree of the polynomial trial functions to increase the accuracy.

$\rightarrow$ However on increase of polynomial degree, complexities may arrive as the approximation may not work within two desired points.

$\rightarrow$ Recall in the class POLYNOMIAL APPROXIMATIONS, we have seen that it is better to use piece-wise lower degree polynomials in smaller domians rather that going for a higher degree polynomial for the entire domian (splines).

→ We will use the same philosophy for the approximate solutions of boundary-value problem while using Rayleigh-Ritz (RR) or Galerkin Weighted Residual method (GWRM). How?

→ That is the entire solution domain $D(x)$ is discretised into small small pieces - called elements - and the RR or GWRM are applied in each of these elements.

→ This is the Finite-Element Method.

Note:

* If the variational functional of a problem is already known in advance (solid mechanics) it is better to use Rayleigh-Ritz FEM.

* If the governing differential equation is known in advance (fluid mechanics) it is better to use Galerkin Weighted Residual FEM.

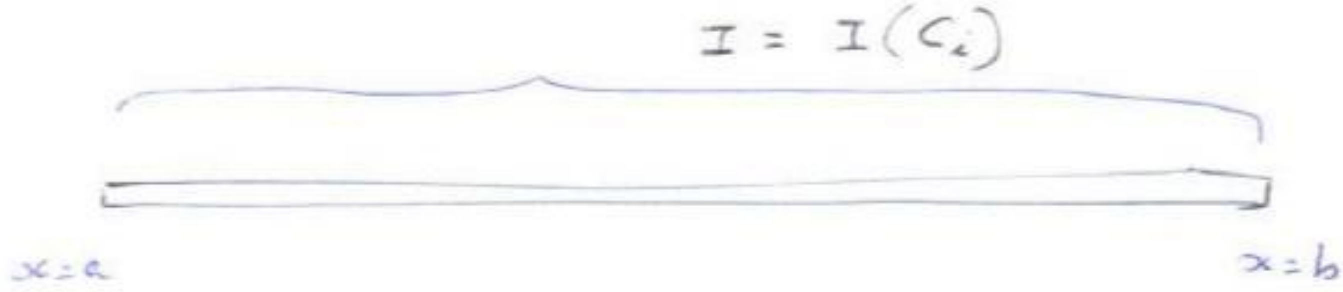
Galerkin FEM

We now describe Galerkin FEM to solve the simple linear BV-ODE

$$\frac{d^2 y}{dx^2} + Qy = F \quad \text{(with appropriate BCs)}$$

The concept involves:

→ The overall outline involves:



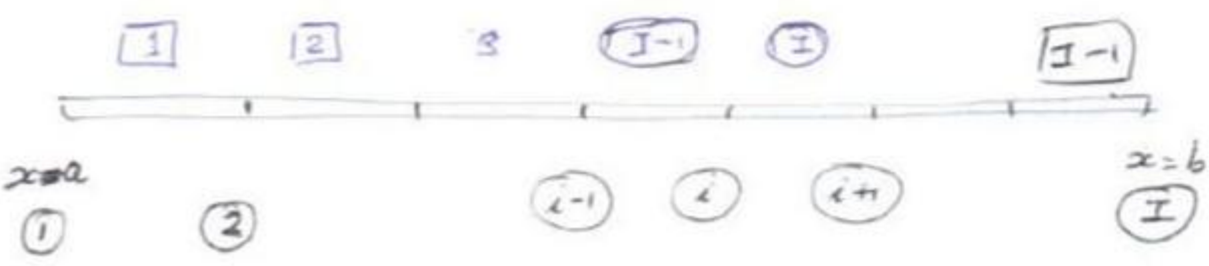
The actual GWRM for the entire domain.

$$\tilde{y} = \sum_{i=1}^{M} C_i y_i(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_M y_M(x)$$

$$\int_a^b W_j(x) R(x) dx = 0 = I(C_j)$$

→ $R(x)$ is evaluated by using $\tilde{y}(x)$.

→ In FEM:



→ The domain is discretised into discrete nodes (1), (2), (3),...,(I)

→ Again the domain consists of $\boxed{I - 1}$ elements

(elements can be linear, quadratic etc.)

$\Delta x^{(i)} = x_{i+1} - x_i$ (element length)

→ Also total integration, $I = I^{(1)} + I^{(2)} + \cdots + I^{(I-1)}$

Summation of integrals for all the elements $1, 2, ..., I-1$.

i.e. $I^{(i)} = \int_{x_i}^{x_{i+1}} W(x) R(x) dx$, integration over an element.

→ On applying this equation for all the weights $W_j(x)$ within the element $\boxed{i}$,

we will get a set of equations relating the nodal values within each element.

Domain discretisation

$D(x)$ into I nodes, $I-1$ elements

$\Delta x^{(i)} = x_{i+1} - x_i$

The exact solution $y(x) \approx \tilde{y}(x)$

$\tilde{y}(x) \rightarrow$ Sum of series of local interpolating polynomials $y^{(i)}(x); i = 1, 2, 3, .., I-1$

These are valid within each element.

$$\tilde{y}(x) = \sum_{i=1}^{I-1} y^{(i)}(x)$$

Local interpolating polynomial $y^{(i)}(x) = y_i N_i^{(i)}(x) + y_{i+1} N_{i+1}^{(i)}(x)$

where $y_i \rightarrow$ Nodal values of y at node i & $y_{i+1} \rightarrow$ Nodal values of y at node $i+1$

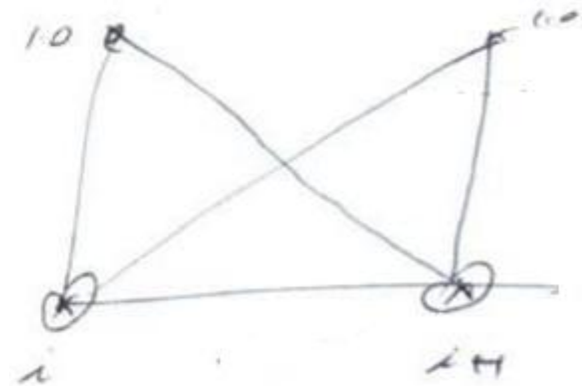
$N_i^{(i)}(x)$ & $N_{i+1}^{(i)}(x) \rightarrow$ Linear interpolating polynomials within element i.

$$N_i^{(i)}(x) = 1.0 \text{ at } x = x_i$$

$$N_i^{(i)}(x) = 0.0 \text{ at } x = x_{i+1}$$

$$N_{i+1}^{(i)}(x) = 0.0 \text{ at } x = x_i$$

$$N_{i+1}^{(i)}(x) = 1.0 \text{ at } x = x_{i+1}$$



$$\therefore y^{(i)}(x) = y_i\left[-\frac{x-x_{i+1}}{\Delta x^{(i)}}\right] + y_{i+1}\left[-\frac{x-x_i}{\Delta x^{(i)}}\right]$$

$N_i \rightarrow$ are called shape functions.

To solve the BV-ODE $\frac{d^2y}{dx^2} + Qy = F$

Define $R(x) = \frac{d^2y}{dx^2} + Qy - F$

$$I(y(x)) = \int_a^b W_j(x)\left[\frac{d^2y}{dx^2} + Qy - F\right]dx = 0 = \int_a^b W_j(x)(y'' + Qy - F)dx$$

$$\int_a^b W_j y''\,dx = -\int_a^b y'W_j'\,dx + \left[y'W_j\right]_a^b = -\int_a^b y'W_j'\,dx + y_b'W_j(b) - y_a'W_j(a)$$

If the Neumann b.c.'s are given: $y_b' = 0, y_a' = 0$

You have $\therefore I(y(x)) = \int_a^b \left[(-y'W_j') + (QyW_j - FW_j)\right]dx = 0$

Again, $I(y(x)) = I^{(1)} + I^{(2)} + \cdots + I^{(I-1)} + y_b'W_I(b) - y_a'W_I(a) = 0$

$$I^{(i)} = \int_{x_i}^{x_{i+1}} \left[-y' \frac{dW_j}{dx} + QyW_j - FW_j \right] dx$$

Aslo, $y^{(i)}(x) = \left[-\frac{x - x_{i+1}}{\Delta x^{(i)}} \right] y_i + \left[\frac{x - x_i}{\Delta x^{(i)}} \right] y_{i+1}$

As in Galerkin method, the weighing functions are same as shape functions $N_i^{(i)}(x)$ and $N_{i+1}^{(i)}(x)$.

$\therefore N_i^{(i)}(x) = 0.0$ for $x > x_{i+1}$ & $x < x_i$

$N_{i+1}^{(i)}(x) = 0.0$ for $x > x_{i+1}$ & $x < x_i$

$$\therefore I^{(i)} = \int_{x_i}^{x_{i+1}} \left[-y' \frac{d}{dx}(N_i^{(i)}(x)) + QyN_i^{(i)} - FN_i^{(i)} \right] dx = 0 \qquad \rightarrow (A)$$

$$\text{Also, } I^{(i)} = \int_{x_i}^{x_{i+1}} \left[-y' \frac{d}{dx}(N_{i+1}^{(i)}(x)) + QyN_{i+1}^{(i)} - FN_{i+1}^{(i)} \right] dx = 0 \rightarrow (B)$$

(A) and (B) are element equations.