CE 601: Numerical Methods

Lecture 35

Finite Element Methods - II

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- In the last class, you have briefly seen the methods that can be used to solve PDEs (or ODEs) other than your finite-difference methods.

$\rightarrow$ For a BV-ODE,

$$\frac{d^2 y}{dx^2} + Qy = F; \; y(x_1) = y_1 \text{ and } y(x_2) = y_2$$

We can suggest the solution in the domain to be

$$y(x) \approx \tilde{y}(x) = \sum_{i=1}^{I} C_i y_i(x)$$

where $y_i(x)$ are trial functions (usually polynomials).

To determine the appropriate coefficients C_i in $\tilde{y}(x)$, we can use

* Rayleigh-Ritz
* Collocation
* Galerkin Weighted Residual

Rayleigh-Ritz Example

- The Rayleigh-Ritz method was briefly explained before. We will see the process through a demonstrative case.

ODE: $\dfrac{d^2 y}{dx^2} + Qy = F;\ y(0.0) = 0.0$ and $y(1.0) = Y.$

$*$ Here is a BV-ODE for the physical domain

from $x = 0.0$ to $x = 1.0.$

$*$ Dirichlet B.C.'s are suggested.

Step (i)

The functional is $I[y(x)] = \int_0^1 G(x, y, y')dx$

The fundamental function is $G(x, y, y') = (y')^2 - Qy^2 + 2Fy.$

We have already seen that the Euler equation for G is giving you the BV-ODE.

Step (ii)

Form the approximate solution for the BV-ODE

$$y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

We are taking three linearly independent polynomials

$$y_1(x) = x, \quad y_2(x) = x(x-1), \quad y_3 = x^2(x-1)$$

$$\therefore \tilde{y}(x) = C_1 x + C_2 x(x-1) + C_3 x^2(x-1) \quad \rightarrow (1)$$

The coefficients C_1, C_2, C_3 are yet unknowns.

$*$ Apply boundary conditions

$\tilde{y}(0.0) = 0.0$

$\tilde{y}(1.0) = C_1 + 0 = Y; \quad \therefore C_1 = Y.$

$\therefore$ Eq.(1) becomes

$$\tilde{y}(x) = Yx + C_2 x(x-1) + C_3 x^2 (x-1) \equiv \tilde{y}(x, C_2, C_3)$$

<u>Step (iii)</u>

To obtain the weights or coefficients C_2 and C_3, we extremise the functional I after substituting in it with approximate $\tilde{y}(x)$.

i.e. $I[y(x)] \approx I[\tilde{y}(x)] = \int_0^1 \left[(\tilde{y}')^2 - Q\tilde{y}^2 + 2F\tilde{y} \right] dx$

Now note

$$\tilde{y}' = Y + (2x-1)C_2 + (3x^2 - 2x)C_3$$

$$I[\tilde{y}(x)] = \int_0^1 \; Y + (2x-1)C_2 + (3x^2-2x)C_3 \;{}^{2}\, dx$$

$$-\int_0^1 Q \;\; Yx + (x^2-x)C_2 + (x^3-x^2)C_3 \;{}^{2}\, dx + \int_0^1 2F \;\; Yx + (x^2-x)C_2 + (x^3-x^2)C_3 \;\; dx$$

$$\Rightarrow I[C_2, C_3]$$

<u>Step (iv)</u>

Extremise the functional

$$\therefore \delta I = 0$$

As $I = I[C_2, C_3]$

$$\therefore \delta I = 0 \text{ requires } \delta I = \frac{\partial I}{\partial C_2}\delta C_2 + \frac{\partial I}{\partial C_3}\delta C_3 = 0$$

We can have this relation only if $\dfrac{\partial I}{\partial C_2} = 0$ and $\dfrac{\partial I}{\partial C_3} = 0.$

As $\tilde{y}' = Y + (2x-1)C_2 + (3x^2-2x)C_3$

$$\therefore \frac{\partial I}{\partial C_2} = \int_0^1 \frac{\partial}{\partial C_2}\left[(\tilde{y}')^2 - Q\tilde{y}^2 + 2F\tilde{y}\right]dx = 0 \quad \rightarrow (2)$$

$$\frac{\partial \tilde{y}'}{\partial C_2} = 2x - 1; \quad \frac{\partial \tilde{y}}{\partial C_2} = x^2 - x; \quad \frac{\partial \tilde{y}'}{\partial C_3} = 3x^2 - 2x; \quad \frac{\partial \tilde{y}}{\partial C_2} = x^3 - x^2.$$

$$\frac{\partial I}{\partial C_3} = \int_0^1 \frac{\partial}{\partial C_3}\left[(\tilde{y}')^2 - Q\tilde{y}^2 + 2F\tilde{y}\right]dx = 0 \quad \rightarrow (3)$$

Substitute the available relations in (2) and (3).

Also do the integration, we will get two equations in C_2 and C_3.

$$C_2 \times \left(\frac{1}{3} - \frac{Q}{30}\right) + C_3 \times \left(\frac{1}{6} - \frac{Q}{60}\right) = -\frac{QY}{12} + \frac{F}{6} \quad \rightarrow (4)$$

$$C_2 \times \left(\frac{1}{6} - \frac{Q}{60}\right) + C_3 \times \left(\frac{2}{15} - \frac{Q}{105}\right) = -\frac{QY}{20} + \frac{F}{12} \quad \rightarrow (5)$$

On solving equations (4) and (5), we get appropriate values of C_2 and C_3.

Subsequently you get the approximate solution:

$$\tilde{y}(x) = Yx + C_2(x^2 - x) + C_3(x^3 - x^2)$$

The Collocation Method

$\rightarrow$ This is a residual method to evaluate solution of a BV-ODE.

$\rightarrow$ For the ODE: $\dfrac{d^2y}{dx^2} + Qy = F;\ y(x_1) = y_1$ and $y(x_2) = y_2$.

$$y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_I y_I(x)$$

$\rightarrow$ Define residual $R(x)$ such that

$$R(x) = \frac{d^2y}{dx^2} + Qy - F$$

For exact solution of $y \rightarrow R(x) = 0$

$\rightarrow$ However as $y \approx \tilde{y}(x)$

$$R(x) = \frac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F = R(x, C_1, C_2, \ldots, C_I)$$

$\rightarrow$ Substitute the residual equation relation

$R(x, C_1, C_2, \ldots, C_I) = 0$ at I distant locations in the domain.

$\therefore$ You get I equatins in $C_1, C_2, \ldots, C_I$.

You can solve for $C_1, C_2, \ldots, C_I$.

In the previous example case:

$$\frac{d^2y}{dx^2} + Qy = F; \; y(0.0) = 0.0 \text{ and } y(1.0) = 1.0.$$

<u>Step (i)</u>: The differential equation is $\frac{d^2y}{dx^2} + Qy = F \quad \rightarrow (1)$

<u>Step (ii)</u>: Assume the approximate solution for ODE (1)

$$y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

Recall in the previous case with boundary conditions applied, we get

$$\tilde{y}(x) = Yx + C_2(x^2 - x) + C_3(x^3 - x^2)$$

Step (iii): Define residual, $R(x) = \dfrac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$

i.e. $\tilde{y}'(x) = Y + C_2(2x-1) + C_3(3x^2 - 2x)$

$\tilde{y}''(x) = 2C_2 + C_3(6x-2)$

$\therefore R(x) = 2C_2 + C_3(6x-2) + Q \times (Yx + C_2(x^2 - x) + C_3(x^3 - x^2)) - F$

i.e. $R(x) = (x^2 - x + 2)C_2 + (Qx^3 - Qx^2 + 6x - 2)C_3 + QYx - F \quad \rightarrow (2)$

There are two unknowns in $R(x) \rightarrow C_2$ and C_3.

Step (iv): Substitute eq. (2) (i.e. $R(x)$) at two points within the domain.

Say $x = 1/3$ and $x = 2/3$.

Write $R(1/3)$ and $R(2/3)$ and solve for C_2 and C_3.

Galerkin Weighted Residual Method

→ This is again a residual method.

As in collocation method,

∗ The approximate solution $\tilde{y}(x)$ used in ODE to get the residual function $R(x)$.

∗ This residual function is weighted using $W_j(x)$ $(j = 1, 2, \ldots)$ and integrated in the entire domain.

∗ Number of weights = No. of unknowns $C_1, C_2, \ldots, C_I$

Any function can be usd as weighting function (Galerkin proved that the trial functions used earlier can act as weighing functions).

In a nutshell

$*$ The required ODE

$*$ Approximate solution formation $\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots$

$*$ Residual definition $R(x) = \dfrac{d^2 \tilde{y}}{dx^2} + Q\tilde{y} - F$

$*$ Weighting residual in domain and integrating

$$\int_a^b W_j(x) R(x) dx = 0$$

$*$ Use that many weights or unknowns present

(two weights if two unknowns)

$*$ Solve the algebraic equations for $C_1, C_2, C_3, \ldots, C_I$.

For the same example case:

Step (i): The ODE: $\dfrac{d^2y}{dx^2} + Qy = F;\; y(0.0) = 0$ and $y(1.0) = Y$.

Step (ii): $y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$

where $y_1(x) = x, y_2(x) = x(x-1), y_3(x) = x^2(x-1)$

Here now you have, $\tilde{y}(x) = Yx + C_2 x(x-1) + C_3 x^2(x-1)$

Step (iii): Residual function $R(x) = \dfrac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$

Recall we evaluated this as

$$R(x) = (x^2 - x + 2)C_2 + (Qx^3 - Qx^2 + 6x - 2)C_3 + QYx - F$$

We have two unknowns C_2 and C_3.

Step (iv): Apply weighted residuals in the domain.

As we require C_2 and C_3

Let us assume $\left.\begin{aligned} W_2(x) &= x^2 - x \\ W_3(x) &= x^3 - x^2 \end{aligned}\right\}$ Preferred weights.

as two weights in $\int_0^1 W_j(x)R(x)dx = 0$

$$\therefore \int_0^1 (x^2 - x)\times\left[(x^2 - x + 2)C_2 + (Qx^3 - Qx^2 + 6x - 2)C_3 + QYx - F\right]dx = 0 \rightarrow (1)$$

$$\int_0^1 (x^3 - x^2)\times\left[(x^2 - x + 2)C_2 + (Qx^3 - Qx^2 + 6x - 2)C_3 + QYx - F\right]dx = 0 \rightarrow (2)$$

Equations (1) and (2) are to be solved for C_2 and C_3.