CE 601: Numerical Methods

Lecture 34

Introduction to FEM

Lecture Delivered By

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Finite Element Methods

- Till the last class, we studied the various FDM used to solve PDES – like elliptic, parabolic, hyperbolic.
- In FDM, the derivatives in PDEs were approximated by difference formulas and subsequently the PDE is converted to algebraic equations at any point in the domain
- There are other numerical methods in addition to FDM to solve partial differential equations.
- Finite-Element Method is one among them.

FEM…

- FEM is a vast topic and itself is a separate course in many curriculum.
- However, as part of numerical methods, we would also like to introduce you the concept of FEM in next few lectures.
- Moreover, FEM is also widely used in many computational programs
- Also it has some advantage over FDM in some cases.
- The objective of FEM as like in FDM is to find the solutions of PDEs and/or ODEs that represent the governing phenomena.
- The solution of the differential equation is given by linear combinations of approximate solutions.

FEM (Contd..)

- We use several trial functions as approximates and subsequently utilize them in various linear combinations to get the approximate solution of PDE.
- These trial functions independently satisfy the boundary conditions of the governing PDE.
- For combinations, different weightage factors or coefficients are assigned.

$\rightarrow$ *e.g.* For a BV-ODE,

$$\frac{d^2 y}{dx^2} + Qy = F; \; y(x_1) = y_1 \text{ and } y(x_2) = y_2.$$

We can suggest the solution in the domain to be

$$y(x) \approx \tilde{y}(x) = \sum_{i=1}^{I} C_i y_i(x)$$

where $y_i(x)$ are trial functions (usually polynomials).

To determine the appropriate coefficients C_i in $\tilde{y}(x)$,

we can use

$*$ Rayleigh-Ritz

$*$ Collocation

$*$ Galerkin Weighted Residual

The Rayleigh-Ritz Method

$\rightarrow$ Based on calculus of variations

$\rightarrow$ Functionals - Function of functions

$\rightarrow$ Objective - Extremise the functionals

Consider the problem

$$\frac{d^2y}{dx^2} + Qy = F;\ y(x=a) = y_a; y(x=b) = y_b.$$

$\rightarrow$ The simplest problem of calculus of variations in one independent variable (x) is $I[y(x)]$:

$$I[y(x)] = \int_a^b G(x, y, y')dx; \quad \left(\text{where } y' \equiv \frac{dy}{dx}\right)$$

where $G(x, y, y') \rightarrow$ is called Fundamental function.

$*$ To extremise the functional $I[y(x)]$, we require $\delta I = 0$.

$*$ Let the two end points $x_1 = a$, and $x_2 = b$.

$$\therefore \delta I = \int_a^b \left(\frac{\partial G}{\partial y} \delta y + \frac{\partial G}{\partial y'} \delta y' \right) dx; \quad \left[\text{Note: } \delta y' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\delta y) \right]$$

$$= \int_a^b \left(\frac{\partial G}{\partial y} \delta y + \frac{\partial G}{\partial y'} \frac{d}{dx} (\delta y) \right) dx$$

Using integration by parts:

$$\int_a^b \left(\frac{\partial G}{\partial y'} \frac{d}{dx} (\delta y) \right) dx = \frac{\partial G}{\partial y'} \delta y \Big|_b^a - \int_a^b \delta y \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) dx$$

$$\text{Now } \delta y = 0 \begin{cases} \text{at } x = a \\ \text{at } x = b \end{cases}$$

$$\therefore \delta I = 0 = \int_a^b \frac{\partial G}{\partial y} \delta y dx - \int_a^b \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) \delta y dx = \int_a^b \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \left(\frac{\partial G}{\partial y'} \right) \right) \delta y dx$$

For non-trivial cases:

$$\boxed{\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0} \qquad \rightarrow (1)$$

This is Euler equation of the calculus of variations.

The meaning of eq. (1) is that

$*$ If a functional $I[y(x)]$ is developed and its extremum (min or max) is zero, then,

$*$ Euler equation is obtained which will be a differential equation whose solution is $y(x)$.

$\rightarrow$ In $\dfrac{d^2y}{dx^2} + Qy = F;\ y(a) = y_a; y(b) = y_b$

$y(x)$ is the actual solution.

Let us form a functional in $y(x)$ i.e. $I[y(x)]$ such that $\delta I = 0$.

$\rightarrow$ The fundamental function is $G(x, y, y')$.

Let $G(x, y, y') = (y')^2 - Qy^2 + 2Fy$

You can see that

$$I[y(x)] = \int_a^b \left[(y')^2 - Qy^2 + 2Fy \right] dx \text{ and } \delta I = 0.$$

Applying Euler equation

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) = 0$$

i.e. $\frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right)$

$$\frac{\partial G}{\partial y} = \frac{\partial}{\partial y}\left[(y')^2 - Qy^2 + 2Fy \right] = 0 - 2Qy + 2F$$

$$\frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) = \frac{d}{dx}\left(\frac{\partial}{\partial y'} \left\{ (y')^2 - Qy^2 + 2Fy \right\} \right) = \frac{d}{dx}\left[2y' - 0 + 0 \right] = 2\frac{d^2 y}{dx^2}$$

$$-2Qy + 2F = 2\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} + Qy = F \rightarrow \text{The said BV-ODE}$$

This means that the function $y(x)$ that extremises the Functional $I[y(x)]$ also satisfies the BV-ODE.

$\therefore$ y in both are same.

<u>Steps in Rayleigh-Ritz</u>

(i) Determine the functional $I[y(x)]$ that can give BV-ODE using Euler equation.

(ii) Assume $y(x) \approx \tilde{y}(x) = \sum_{i=1}^{I} C_i y_i(x)$

$y_i(x)$ are trial functions.

(iii) Substitute $\tilde{y}(x)$ in I

We will get $I[C_1 y_1 + C_2 y_2 + \cdots + C_I y_I]$

Please note $y_1, y_2, \ldots, y_I$ are polynomials in x.

$\therefore$ For $\delta I = 0$, we get

$$\frac{\partial I}{\partial C_1} = 0, \frac{\partial I}{\partial C_2} = 0, \cdots, \frac{\partial I}{\partial C_I} = 0.$$

(iv) Solve for the coefficients C_i.

For a specific example

$$\frac{d^2 y}{dx^2} + Qy = F; \; y(0.0) = 0.0, y(1.0) = Y$$

Soln. (i) The functional $I[y(x)]$

$$I[y(x)] = \int_0^1 \left[(y')^2 - Qy^2 + 2Fy \right] dx;$$

So, $G(x, y, y') = (y')^2 - Qy^2 + 2Fy.$

You have already seen that $\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$ gave the ODE.

(ii) $y(x) \approx \tilde{y}(x) = C_1 y_1 + C_2 y_2 + C_3 y_3$

$y_1(x), y_2(x)$ and $y_3(x)$ are three linearly independent trial functions.

$y_1(x) = x, y_2(x) = x(x-1), y_3(x) = x^2(x-1).$

Please note our boundary on right is $x = 1.0$.

$\therefore \tilde{y}(x) = C_1 x + C_2 x(x-1) + C_3 x^2 (x-1)$

We know at $x = 0, y = 0$

at $x = 1, y = Y$

i.e. $Y = C_1 + 0,$ i.e. $C_1 = Y.$

$\therefore \tilde{y}(x, C_2, C_3) = Yx + C_2 x(x-1) + C_3 x^2 (x-1)$

(iii) $I[\tilde{y}(x)] = \int_0^1 \left[(\tilde{y}')^2 - Q\tilde{y}'^2 + 2F\tilde{y}' \right] dx = I[C_2, C_3].$

(iv) For $\delta I = 0$, we have now

$$\frac{\partial I}{\partial C_2} = 0 = \int_0^1 \frac{\partial}{\partial C_2}(\tilde{y}'^2)dx - \int_0^1 \frac{\partial}{\partial C_2}(Q\tilde{y}^2)dx + \int_0^1 \frac{\partial}{\partial C_2}(2F\tilde{y})dx$$

$$\tilde{y}' = Y + C_2(2x-1) + C_3(3x^2 - 2x)$$

$$\therefore \frac{\partial \tilde{y}'}{\partial C_2} = 2x-1; \frac{\partial}{\partial C_2}(\tilde{y}'^2) = 2C_2(x^2 - x)^2 + 2(x^3 - x^2)(x^2 - x); \frac{\partial \tilde{y}}{\partial C_2} = x(x-1).$$

$$\frac{\partial I}{\partial C_3} = 0 = \int_0^1 \left[\frac{\partial}{\partial C_3}(\tilde{y}'^2) - \frac{\partial}{\partial C_3}(Q\tilde{y}^2) + \frac{\partial}{\partial C_3}(2F\tilde{y}) \right] dx$$

$$\frac{\partial \tilde{y}}{\partial C_3} = x^3 - x^2; \frac{\partial \tilde{y}'}{\partial C_3} = 3x^2 - 2x$$

Substitute and integrate you will get two equations to solve C_2 and C_3.

These equations are:

$$C_2 \times \left(\frac{1}{3} - \frac{Q}{30} \right) + C_3 \times \left(\frac{1}{6} - \frac{Q}{60} \right) = -\frac{QY}{12} + \frac{F}{6}$$

$$C_2 \times \left(\frac{1}{6} - \frac{Q}{60} \right) + C_3 \times \left(\frac{2}{15} - \frac{Q}{105} \right) = -\frac{QY}{20} + \frac{F}{12}$$

Once you get C_1 and C_2

$$\tilde{y}(x) = Yx + C_2 x(x-1) + C_3 x^2 (x-1).$$