CE 601: Numerical Methods

Lecture 33

Finite Difference Methods for PDEs

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- We have discussed the following finite-difference methods to solve **Parabolic PDEs.**
  - Forward-time Centered Space (FTCS)
  - Backward-time Centered Space (BTCS)
  - Crank-Nicholson method (where $O(\Delta t^2)$ and $O(\Delta x^2)$ for derivatives are used).

The next important method is:

- **Alternating Direction Implicit Method (ADI) for Multi-spatial dimension Problems**
- If you have multi-spatial dimension problems say

$$\frac{\partial f}{\partial t} = \alpha \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right]$$

- If you use FTCS, then the FDE will be:

$$\boxed{f_{i,j}^{(n+1)} = f_{i,j}^{(n)} + \alpha \Delta t . \left[\frac{f_{i-1,j}^{n} - 2.f_{i,j}^{n} + f_{i+1,j}^{n}}{\Delta x^2} + \frac{f_{i-1,j}^{(n)} - 2.f_{i,j}^{(n)} + f_{i+1,j}^{(n)}}{\Delta y^2} \right]} \rightarrow (1)$$

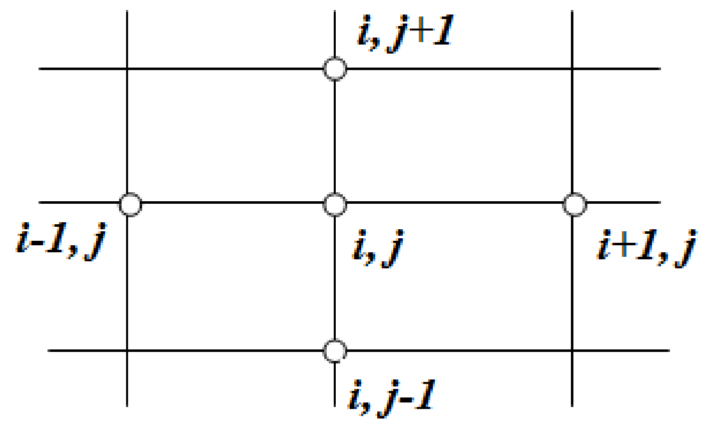
- Similarly, you can frame fully implicit FDE using BTCS, i.e.

$$\left[-\frac{\alpha\Delta t}{\Delta x^2}\right] f_{i-1,j}^{(n+1)} + \left[-\frac{\alpha\Delta t}{\Delta y^2}\right] f_{i,j-1}^{(n+1)} + \left[1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right] f_{i,j}^{(n+1)} + \left[-\frac{\alpha\Delta t}{\Delta y^2}\right] f_{i,j+1}^{(n+1)} + \left[-\frac{\alpha\Delta t}{\Delta x^2}\right] f_{i+1,j}^{(n+1)} = f_{i,j}^{(n)} \rightarrow (2)$$

- On applying eq.(2) at each grid node *(i,j)*, we will get a system of linear equation. This system will be here banded and pentadiagonal. You can use appropriate solvers to solve each banded matrix system.

- However the implicit technique is computationally tedious.
- We can go for Alternate-Direction Implicit Method (ADI).
- This method involves two steps:
- Recall earlier we said

$$\left.\frac{\partial f}{\partial t}\right|_{i,j}^{n+\frac{1}{2}} \simeq \frac{1}{2}\left[\left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n+1)} + \left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n)}\right]$$



- That is this quantity consist of half of implicit quantity and half of explicit quantity.

- Since $\dfrac{\partial f}{\partial t} = \alpha\left[\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right]$
- For

$$\left.\frac{\partial f}{\partial t}\right|_{i,j}^{n+1/2} = \frac{1}{2}\left[\left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n+1)} + \left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n)}\right]$$

$$= \alpha\left[\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j}^{(n+1)} + \left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j}^{(n)}\right]$$

- i.e. Step-1

$$\frac{f_{i,j}^{(n+1)} - f_{i,j}^{(n)}}{\Delta t} = \alpha.\left[\frac{f_{i-1,j}^{(n+1)} - 2.f_{i,j}^{(n+1)} + f_{i+1,j}^{(n+1)}}{\Delta x^2} + \frac{f_{i-1,j}^{(n)} - 2.f_{i,j}^{(n)} + f_{i+1,j}^{(n)}}{\Delta y^2} \right]$$

- Step-2

$$\left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n+1+1/2)} = \frac{1}{2}\left[\left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n+1)} + \left.\frac{\partial f}{\partial t}\right|_{i,j}^{(n+2)} \right] = \alpha \left[\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j}^{(n+1)} + \left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j}^{(n+1)} \right]$$

i.e. $$\frac{f_{i,j}^{(n+2)} - f_{i,j}^{(n+1)}}{\Delta t} = \alpha.\left[\frac{f_{i-1,j}^{(n+1)} - 2.f_{i,j}^{(n+1)} + f_{i+1,j}^{(n+1)}}{\Delta x^2} + \frac{f_{i-1,j}^{(n+2)} - 2.f_{i,j}^{(n+2)} + f_{i+1,j}^{(n+2)}}{\Delta y^2} \right]$$

- So using these two steps ADI is implemented in the numerical scheme.

Hyperbolic PDE

- We have seen before that hyperbolic PDEs are also
  - Propagation problems
  - We have to use marching methods to solve hyperbolic PDEs.
  - It has definite (finite) propagation speed for the physical information propagation.
  - You have distinct domain of dependence and range of influence.
- Some examples of hyperbolic PDE:

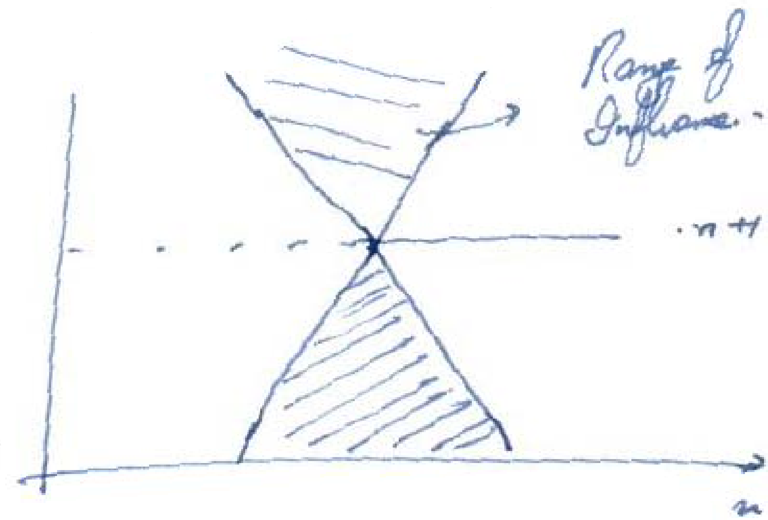
$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \text{ (The Convection Equation)}$$

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2} \text{ (The Wave Equation)}$$

- The convection equation is usually used in
  - Fluid mechanics
  - Heat transfer
- Wave equation is used for
  - Vibrating systems (strings, acoustic fields)
- As usual we can have
  - Dirichlet B.C.
  - Neumann B.C.
  - Mixed B.C.

for various hyperbolic PDEs.

- Recall as hyperbolic PDE has finite range of influence as well as domain of dependence.
- The finite physical propagation speed is approximated as



$$c_n = \frac{\Delta x}{\Delta t}$$

Actual is $\frac{dx}{dt}$.

For $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0;$ we have $\frac{dx}{dt} = u$

For $\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2},$ we have $\frac{dx}{dt} = \pm c$

Convection Equation

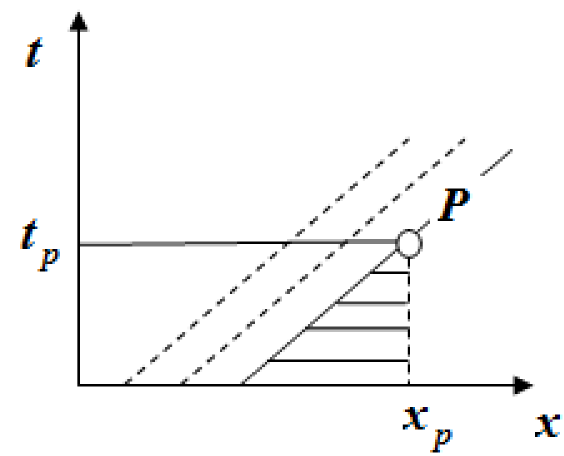
- The derivative $\frac{df}{dt}$ in the convection equation can be approximated using

  -> Forward Time or

  -> Backward Time or

  -> Centered Time

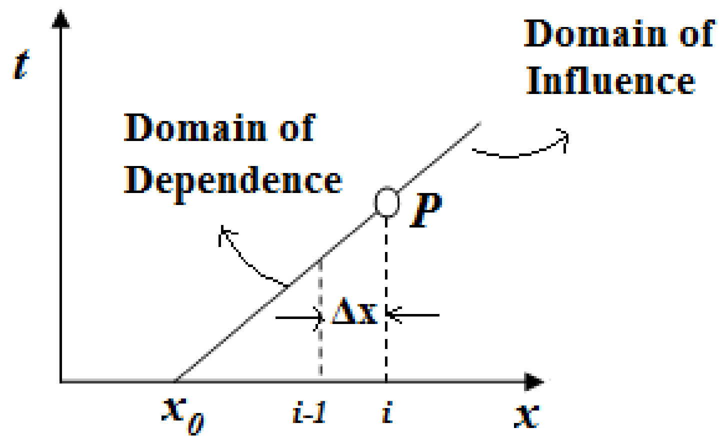
  while using FDM.



- In $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0;$
- $\frac{\partial f}{\partial x}$ suggest something on physical convection.
- The physical information propagation speed is

$$\frac{dx}{dt} = u$$

- The solution at a point depends only on the information in the domain of dependence specified by upstream characteristic paths.



- The first derivative $\frac{\partial f}{\partial x}$ (spatial) should be approximated by one-sided approximations in the direction from which physical information is propagated. They are **upwind** approximations.

$$\therefore \left.\frac{\partial f}{\partial x}\right|_i \approx \frac{f_i - f_{i-1}}{\Delta x}$$

We can also use centered space approximations with acceptable results. i.e.

$$\left.\frac{\partial f}{\partial x}\right|_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

•Upwind Methods

- For the convection equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

$$\frac{dx}{dt} = u$$

- Information propagates with
- If $u > 0$, information propagates from left to right.
- If $u < 0$, information propagates from right to left.

- First order upwind scheme: For $u > 0$, i.e.

- $$\left.\frac{\partial f}{\partial t}\right|_i^{(n)} + u.\left.\frac{\partial f}{\partial x}\right|_i^{(n)} = 0;$$

i.e. $$\frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} + u \cdot \frac{f_i^{(n)} - f_{i-1}^{(n)}}{\Delta x} = 0$$

or, $$\boxed{f_i^{(n+1)} = f_i^{(n)} - c.\left(f_i^{(n)} - f_{i-1}^{(n)}\right)}$$

where $c = \dfrac{u\Delta t}{\Delta x}$ $\rightarrow$ Convection Number

From the stability point of view, you require $c \leq 1.0$.

- This is because the actual speed $$u = \frac{dx}{dt}.$$
- Numerical speed $C_n = \frac{\Delta x}{\Delta t}.$
- The convection number suggest that you require atleast a certain computational speed c_n that is not less than the actual convection speed.
- Or, if the actual speed exceeds numerical speed, results fluctuates.
- This again boils down to the criteria that you cannot have large values of Δt. If Δt is large, then computational convection speed reduces.