CE 601: Numerical Methods

Lecture 32

PDE Characteristics & FDM

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- We have already discussed about the characteristics of PDE for both quasi-linear second order PDE as well as quasi linear first order PDE.
- Characteristics are paths through which information is propagated.
- For two-dimensional solution domain *D(x,y)* of the quasi-linear second order PDE, the characteristics are given by the slope

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

- For two-dimensional solution domain *D(x,t)* of the quasi-linear first order PDE, the slope (or characteristic)

$$\frac{dx}{dt} = \frac{b}{a}$$

- The parabolic PDE: $\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$
- Here, the solution domain is $D(x,t)$.
- From the characteristic analysis, we have

$$\begin{bmatrix} \alpha & 0 & 0 \\ dx & dt & 0 \\ 0 & dx & dt \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x \partial t} \\ \frac{\partial^2 f}{\partial t^2} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial f}{\partial t} \\ d\left(\frac{\partial f}{\partial x}\right) \\ d\left(\frac{\partial f}{\partial t}\right) \end{Bmatrix}$$

- Hence, the characteristic equation will be

$$\alpha \left(dt\right)^2 = 0 \Rightarrow dt = \pm 0$$

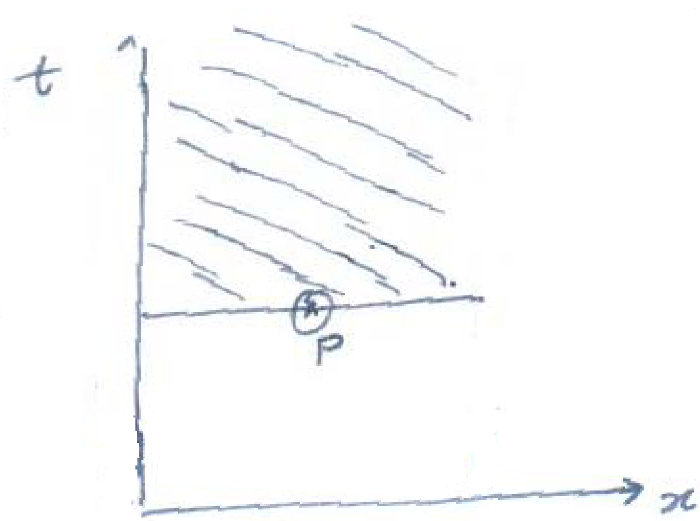
- So, t is a constant.
- Hence, the characteristics are lines of constant time.
- We can see that there are two real, repeated roots associated with characteristic equation

$$\alpha dt^2 = 0$$

- Speed of propagation of information along these characteristic paths can be described as such:

$$\text{Speed} = c = \frac{dx}{dt} = \frac{dx}{0} = \pm\infty$$

- So, information propagates at infinite speed along lines of constant time.
- Both characteristics have zero slope in the x-t plane (Infinite information propagation).



- Solution at point ‘P’ depends on entire solution domain below including horizontal line of P.
- Solution at P influences entire solution domain above including horizontal time line of P.
- When we use numerical methods, the infinite information propagation speed should be taken into account.

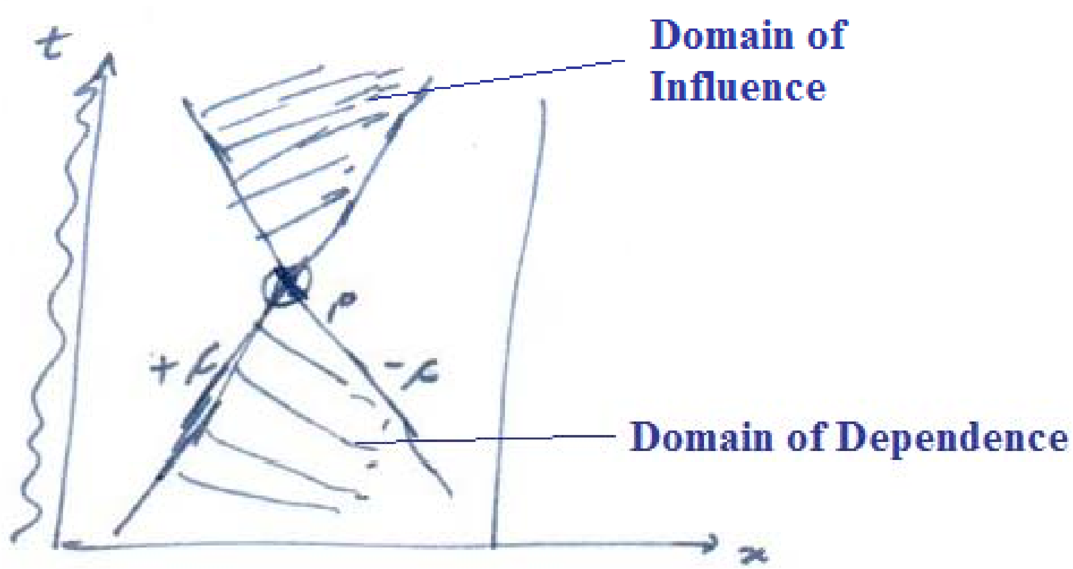
- For hyperbolic PDEs: $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$

- Writing the equation as $\dfrac{\partial^2 f}{\partial t^2} - c^2 \dfrac{\partial^2 f}{\partial x^2} = 0$

- We have $\dfrac{dx}{dt} = \dfrac{0 \pm \sqrt{0 + 4c^2}}{2} \Rightarrow \dfrac{dx}{dt} = \pm c$

- That is, for hyperbolic equation, the information propagates at a finite speed of $\pm c$.

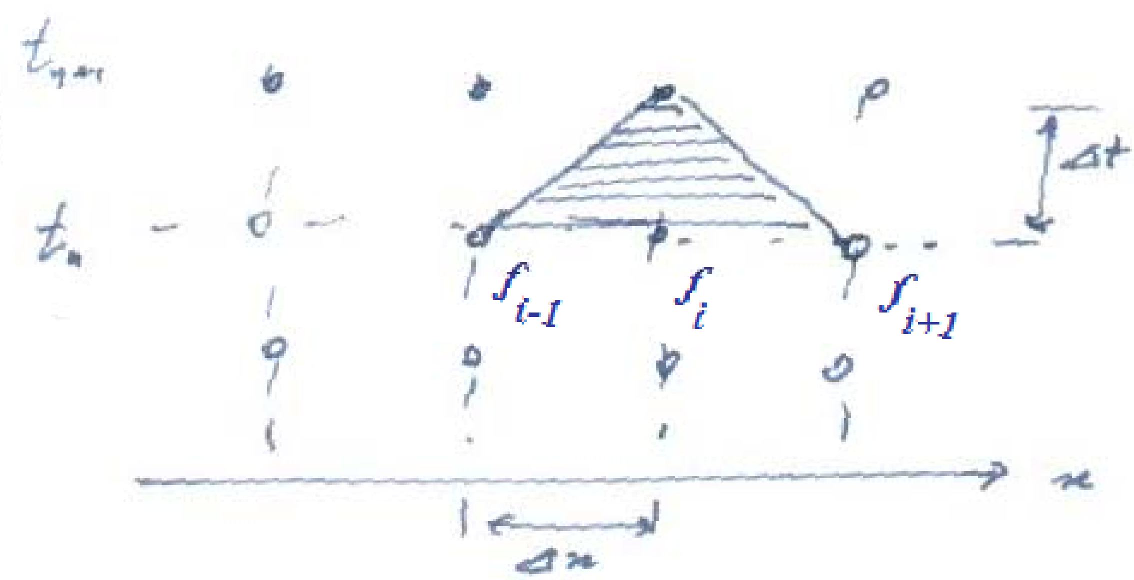
- These characteristics have finite information propagation speed
  - Finite domain of dependence
  - Finite range of influence
- You have seen the FDM applications to elliptic PDEs.



- **<u>Parabolic PDE</u>**:
- They are propagation problems.
- Consider the diffusion equation:

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

- There are semi-infinite domain of dependence and semi-infinite range of influence for parabolic PDEs.
- Hence, we need to adopt marching methods to solve parabolic PDEs.



- March the solution at time level $'n'$ to time level $'n+1'$ i.e. using the information at time t_n, the properties at t_{n+1} can be identified.

- Forward Time Centered Space (FTCS) Method:
- Here, the technique is explicit, the infinite propagation speed is approximated by finite propagation speed $C_n = \frac{\Delta x}{\Delta t}$

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \quad \text{(Parabolic PDE)}$$

$$\left.\frac{\partial f}{\partial t}\right|_i^{(n)} = \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n)} = \frac{f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)}}{\Delta x^2}$$

- $$\frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} = \alpha . \frac{f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)}}{\Delta x^2}$$
- $$f_i^{(n+1)} = f_i^{(n)} + \frac{\alpha \Delta t}{\Delta x^2}\left(f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)} \right)$$
- We need to check for the consistency, stability, order etc.
- We can define diffusion number as $d = \dfrac{\alpha \Delta t}{\Delta x^2}$.

$$\therefore f_i^{(n+1)} = f_i^{(n)} + d.\left(f_{i+1}^{(n)} - 2.f_i^{(n)} + f_{i-1}^{(n)} \right)$$

- Explicit FTCS method needs to be checked for stability.

$$f_i^{(n+1)} = G.f_i^{(n)}; \quad G \Rightarrow \text{Amplification factor}$$

$$\left|G\right| \leq 1.0 \text{ for stability.}$$

- <u>Backward Time Centered Space (BTCS):</u>
- The BTCS formulation takes into account infinite propagation speed.

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

$$\left.\frac{\partial f}{\partial t}\right|_i^{(n+1)} = \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1)} = \frac{f_{i+1}^{(n+1)} - 2.f_i^{(n+1)} + f_{i-1}^{(n+1)}}{\Delta x^2}$$

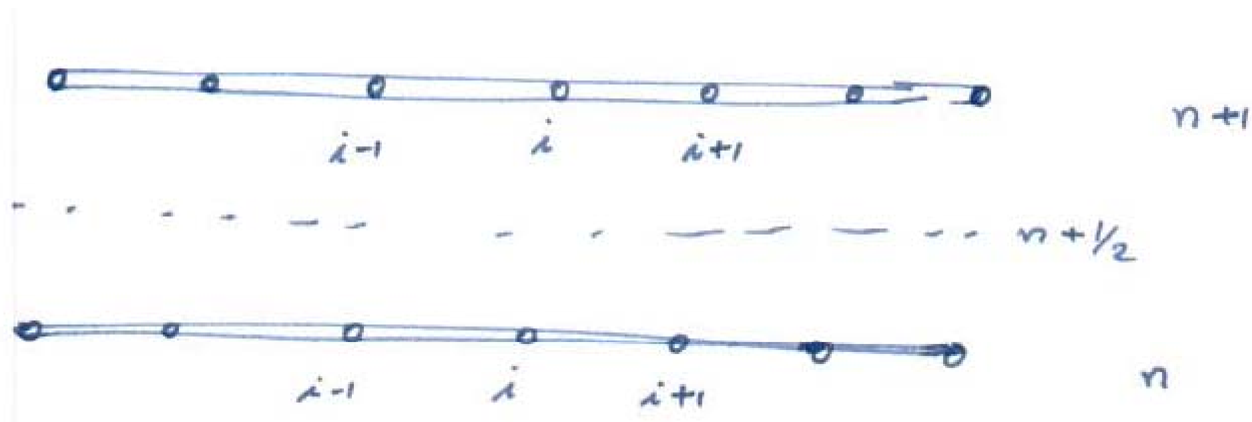
$$\therefore \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} = \alpha . \frac{f_{i+1}^{(n+1)} - 2.f_i^{(n+1)} + f_{i-1}^{(n+1)}}{\Delta x^2}$$

$$\Rightarrow f_i^{(n)} = -\frac{\alpha \Delta t}{\Delta x^2} f_{i-1}^{(n+1)} + \left(1 + \frac{2.\alpha \Delta t}{\Delta x^2}\right) f_i^{(n+1)} - \frac{\alpha \Delta t}{\Delta x^2} f_{i+1}^{(n+1)}$$

- <u>Crank-Nicholson Method</u>
- To solve the diffusion equation:

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

- The temporal and spatial domains are discretized with intervals Δt and Δx respectively.
- Now,



- If we try to evaluate

$$\left.\frac{\partial f}{\partial t}\right|_i^{(n+1/2)} = \alpha.\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1/2)}$$

- How will we evaluate $\left.\frac{\partial f}{\partial t}\right|_i^{(n+1/2)}$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1/2)}$?

- From our earlier discussion, we can easily write $\left.\frac{\partial f}{\partial t}\right|_i^{(n+1/2)} = \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t}; \quad O\left(\Delta t^2\right)$

- Now, we can also write

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1/2)} = \frac{1}{2}\left\{\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n)} + \left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1)}\right\}$$

$$= \frac{1}{2}\left\{\frac{f_{i-1}^{(n)} - 2.f_i^{(n)} + f_{i+1}^{(n)}}{\Delta x^2} + \frac{f_{i-1}^{(n+1)} - 2.f_i^{(n+1)} + f_{i+1}^{(n+1)}}{\Delta x^2}\right\}; O(\Delta x^2)$$

i.e. $$\frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} = \frac{\alpha}{2}\left\{\frac{f_{i-1}^{(n)} - 2.f_i^{(n)} + f_{i+1}^{(n)}}{\Delta x^2} + \frac{f_{i-1}^{(n+1)} - 2.f_i^{(n+1)} + f_{i+1}^{(n+1)}}{\Delta x^2}\right\}$$

As we know diffusion number, $d = \dfrac{\alpha \Delta t}{\Delta x^2}$.

$$\therefore \boxed{\left(-\frac{d}{2}\right) f_{i-1}^{(n+1)} + (1+d)\, f_i^{(n+1)} + \left(-\frac{d}{2}\right) f_{i+1}^{(n+1)} = \left(\frac{d}{2}\right) f_{i-1}^{(n)} + (1-d)\, f_i^{(n)} + \left(\frac{d}{2}\right) f_{i+1}^{(n)}}$$