CE 601: Numerical Methods

Lecture 31

The Classification of PDEs

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The Classification of PDEs

- We discussed about the classification of PDEs for a quasi-linear second order non-homogeneous PDE

$$\boxed{A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x \partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff = G}$$

  as elliptic, parabolic and hyperbolic.
- Such Classification helps in knowing the allowable initial and boundary conditions to a given problem.
- It also helps in the effective choice of numerical methods.

- We can also have another general case: First order quasi-linear non-homogeneous PDE in two independent variables:

$$\boxed{a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=c}$$

- We can also have cases involving two general quasi-linear first order non-homogeneous PDE in two independent variables

$$\boxed{\begin{aligned} &a\frac{\partial f}{\partial t}+b\frac{\partial f}{\partial x}+c\frac{\partial g}{\partial t}+d\frac{\partial g}{\partial x}=e \\ &A\frac{\partial f}{\partial t}+B\frac{\partial f}{\partial x}+C\frac{\partial g}{\partial t}+D\frac{\partial g}{\partial x}=E \end{aligned}}$$

- These are some of the general PDEs that are extensively used.
- The classification of all the above PDEs are related to the characteristics of PDE.
- What are characteristics of PDE?
- If we consider all the independent variables in a PDE as part of describing the domain of the solution than they are dimensions
- e.g. In $$\frac{\partial f}{\partial x} = \alpha \frac{\partial^2 f}{\partial x^2}; f(x,t)$$

The solution '*f*' is in the solution domain *D(x,t).* There are two dimensions *x* and *t*.

- E.g. $\frac{\partial f}{\partial t} = \alpha_1 \frac{\partial^2 f}{\partial x^2} + \alpha_2 \frac{\partial^2 f}{\partial y^2}$

  $f(x, y, t) \rightarrow D(x, y, t) \rightarrow 3 \text{ dimensions}$

  $\frac{\partial f}{\partial t} = \alpha_1 \frac{\partial^2 f}{\partial x^2} + \alpha_2 \frac{\partial^2 f}{\partial y^2} + \alpha_3 \frac{\partial^2 f}{\partial z^2}$

  $f(x, y, z, t) \rightarrow D(x, y, z, t) \rightarrow 4 \text{ dimensions}$

- Similarly higher dimensions can be present.
- Characteristics are (n-1) dimensional hypersurfaces in n-dimensional domain that have special features.
  - ❖ The characteristics are paths in the solution domain along which information propagates.
  - ❖ Discontinuities in derivatives of the dependent variable if it is there, also propagates through these characteristic paths.

- Consider the simple convection problem (say convection of a property '*f*' of a fluid particle) in one spatial dimension:

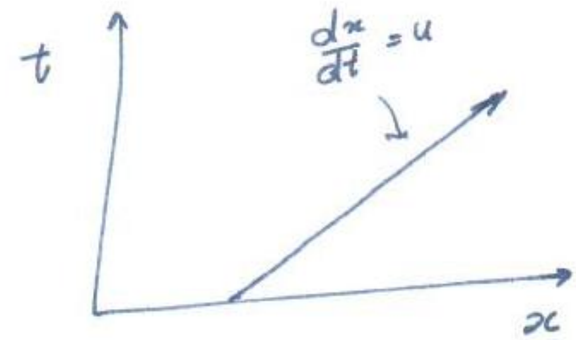
$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

- We can see that the solution domain is *D(x,t)* of two dimensions. Therefore we should try to infer in a domain *(x,t)* whether there are any characteristics paths.
- Let the convection velocity '*u*' be given as *u(t).* A moving fluid particle can carry mass, momentum, energy as it moves through the space. Let convection velocity for that particle be '*u*'.

$$\therefore \frac{dx}{dt} = u \text{ for the particle}$$

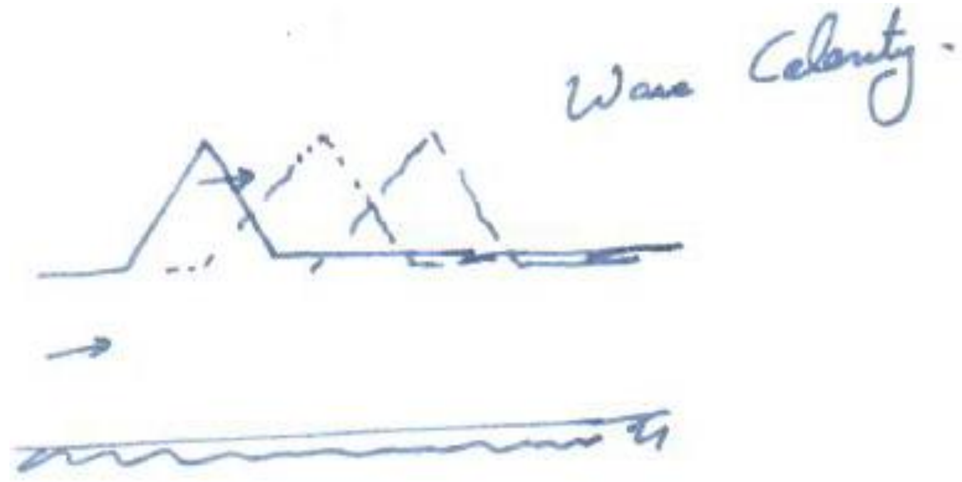
- Or, $x = x_0 + \int_{t_0}^{t} u(t)dt$. This is the pathline of the said fluid particle.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\cdot\frac{\partial f}{\partial x} = 0 = \frac{df}{dt}$$



- On integration, it means that '*f*' will be a constant.
- This means that the property '*f*' is convected along a pathline.
- This pathline becomes the characteristic path for the property '*f*' in the solution domain *(x,t)*.

- In hydrology, some of you might have studied Kinematic Wave Routing. In that characteristic paths are used to rout incoming flood hydrograph.



To determine Characteristics

- Note that discontinuities in the derivatives of the solution (if exist) must propagate along the characteristics.
- We already have the quasi-linear second order PDE:

$$A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}+D\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial y}+Ff=G \quad \rightarrow (1)$$

- If the solution domain is *D(x,y)* for the dependent variable *f(x,y),* then at any general point '*P'* in the solution domain, if $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x\partial y}$ or $\frac{\partial^2 f}{\partial y^2}$ are multi-valued or discontinuous and if a path passes through this general point '*P',* then it is a characteristic path.

- We can use chain rule

$$\left.\begin{aligned} d\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial y}dy \\ d\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)dy \end{aligned}\right\} \rightarrow (2)$$

$$\text{i.e.} \begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 f}{\partial x^2} \\ \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y^2} \end{Bmatrix} = \begin{Bmatrix} -D\dfrac{\partial f}{\partial x} - E\dfrac{\partial f}{\partial y} - Ff - G \\ d\left(\dfrac{\partial f}{\partial x}\right) \\ d\left(\dfrac{\partial f}{\partial y}\right) \end{Bmatrix} \rightarrow (3)$$

- In this system if we set the determinant of coefficient matrix = 0

$$\text{i.e. } A \; dy^{2} - B \; dx \;\; dy \; + C \; dx^{2} = 0$$

$$\text{i.e. } A\left(\frac{dy}{dx}\right)^{2} - B\left(\frac{dy}{dx}\right) + C = 0 \quad \rightarrow (4)$$

- Eq.(4) becomes the characteristic equation

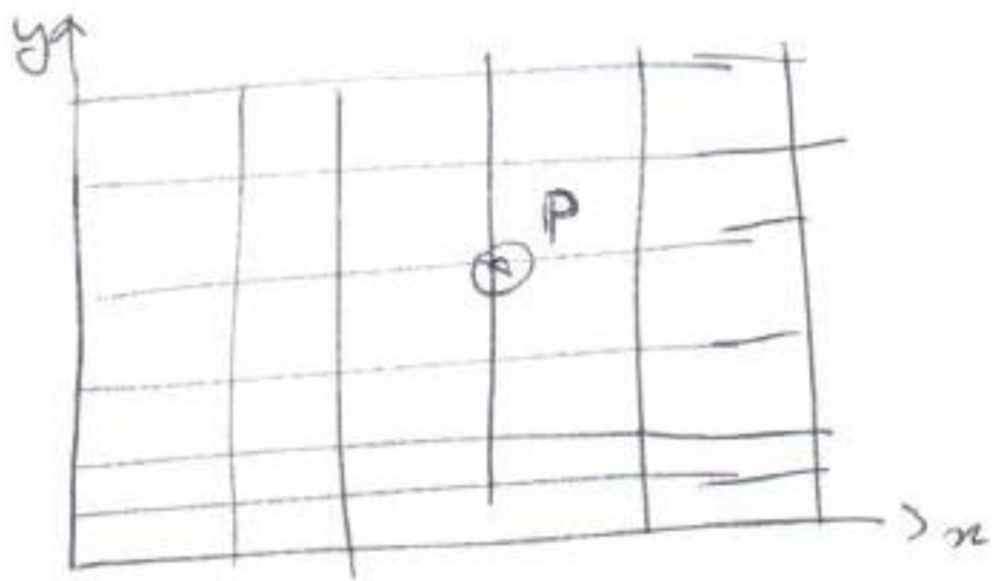
$$\therefore \left(\frac{dy}{dx}\right) = \frac{B \pm \sqrt{B^{2} - 4AC}}{2A}$$

- If $B^{2} - 4AC < 0$, then we have complex characteristic curves (Elliptic PDE).
- If $B^{2} - 4AC = 0$, then we have equal characteristic curves (Parabolic PDE).

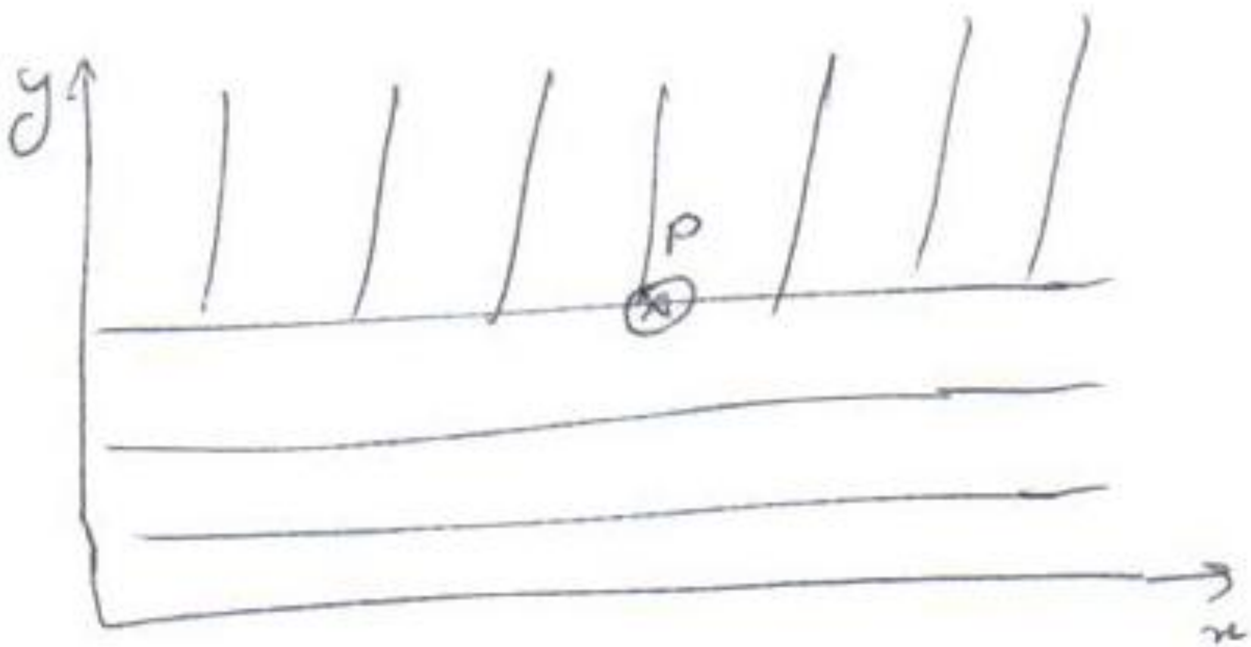
- If $B^2 - 4AC > 0$, then we have hyperbolic PDE and distinct characteristic paths.
  - $\Rightarrow$ Elliptic PDEs have no real characteristic paths.
  - $\Rightarrow$ Parabolic PDEs have one real repeated characteristic path.
  - $\Rightarrow$ Hyperbolic PDEs have two real and distinct characteristic paths.
- Due to presence of characteristic paths in the solution domain say *D(x,y)*, we have
  – Domain of dependence
  – Range of influence

Note in the figures we represent: Horizontal lines as Domain of dependence; Vertical lines as Range of influence.

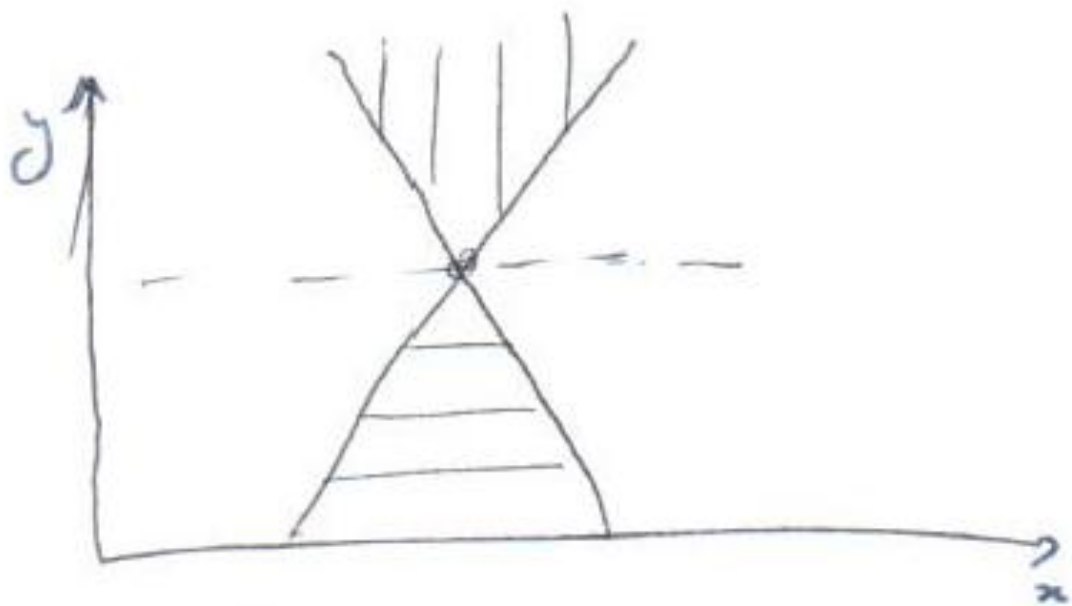
- Elliptic PDE:



- Parabolic PDE



- Hyperbolic PDE



- Consider the point *P* in the solution domain.
- $f(x_p, y_p)$ depends on everything that has happened in domain of dependence.

- For a **quasi-linear** first order non-homogeneous PDE, the PDE is always hyperbolic

$$a\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} = c \quad \rightarrow (1)$$

- The characteristic paths are determined by

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx \quad \rightarrow (2)$$

$$\text{i.e.}\begin{bmatrix} a & b \\ dt & dx \end{bmatrix}\begin{Bmatrix} \partial f/\partial t \\ \partial f/\partial x \end{Bmatrix}=\begin{Bmatrix} c \\ df \end{Bmatrix}$$

- Characteristic equation: $adx - bdt = 0$

$$\text{i.e.}\ \boxed{\frac{dx}{dt}=\frac{b}{a}} \quad \rightarrow (3)$$

- Eq.(3) is differential equation for a family of paths in the solution domain along which $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ may be discontinuous or multi-valued.
- As *'a'* and *'b'* are real valued, real characteristic paths always exist.

- <u>Speed of Propagation</u>
- Speed of propagation of information depends on the slope of the characteristics.
- Physical problems governed by PDEs that have real characteristics are <u>Propagation Problems</u>.

- For elliptic PDE: $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$

$$\therefore \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{0 \pm \sqrt{0-4}}{2} = \pm\sqrt{-1}$$

- Roots are complex.
- Elliptic PDEs are not having any real characteristics.
- The entire solution domain *D(x,y)* is the domain of dependence as well as domain influence.