CE 601: Numerical Methods
Lecture 3

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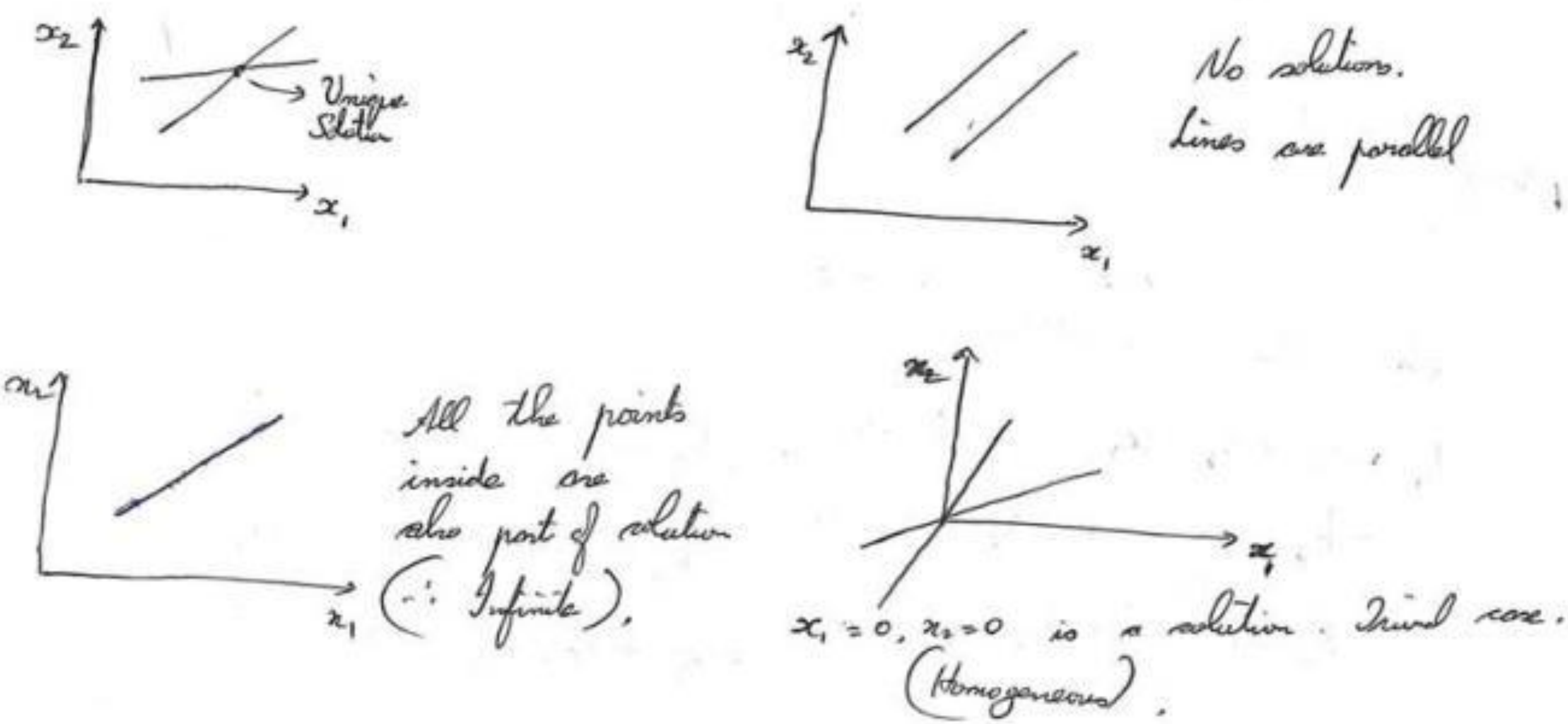
System of Linear Algebraic Equations

- Q. What are the ways to solve the system of linear equations? What are the types of solutions expected from a linear system?

- E.g. consider a linear system:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- The above system represents two straight lines. Therefore we can have following types of solutions.



- Q. How do you solve such system of linear equations?
- There are two approaches:
    - Direct elimination methods
    - Iterative methods

- <u>The Direct Elimination:</u>
- As the name suggests the methods are having procedures of algebraic elimination of the contents in the coefficient matrix that lead to solution.
    - Gauss elimination
    - Gauss-Jordan
    - Matrix inverse
    - LU factorization etc.

- In iterative methods, initially a solution is assumed and through iterations the actual solution is approached asymptotically.
    - Jacobi iteration
    - Gauss-Seidel iteration
    - Successive over relaxation
- <u>Matrix Properties:</u>
- We have seen earlier the system of linear equations can be represented by matrix methods.

- Q. What is a matrix?
- It is an array of elements that are arranged in orderly rows and columns.

$$[A] = \underset{n \times m \text{ matrix}}{\left[a_{ij} \right]} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

- Vectors: Column vector $x = x_i = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{Bmatrix}$

Row vector $y = \left[y_j \right] = \; y_1 \quad y_2 \quad . \quad . \quad . \quad y_m$

- Unit vector -> The vector whose magnitude is 1.

$$\hat{i} = i = \begin{Bmatrix} i_1 \\ i_2 \\ . \\ . \\ i_n \end{Bmatrix} \text{ and } \sqrt{i_1^{\,2} + i_2^{\,2} + \ldots + i_n^{\,2}} = 1$$

- You know what is meant by
  - Square matrix
  - Diagonal matrix
  - Identity matrix
  - Triangular matrix: 1) Upper and 2) Lower

- Also recall that: Matrix addition and Matrix multiplication.
- As a reading exercise please find the properties of matrix: 1. Associative, 2. Commutative and 3. Distributive.
- Square matrices -> properties

- You have if [A] is a n x n matrix,

  then $[A][A]^{-1} = I$

  or, $[A]^{-1}[A] = I$

  if there are two matrices [A] and [B] such that

  $[A][B] = I$ then $[A] = [I]\,[B]^{-1}$
- <u>Matrix Factorization</u>
- A matrix can be represented as product of two other matrices [A]=[B][C]

- <u>For a system of linear algebraic equations</u>

$$A \quad x = b$$

$$\sum_{j=1}^{n} a_{i,j} x_i = b_i; \quad i = 1, 2, 3, \ldots, n.$$

- We can do three row operations on such a linear system that will not alter the solution
    - Scaling
    - Pivoting
    - Elimination
- These row operations are extensively used in eliminations methods.

- <u>Direct Elimination Method</u>
- To perform elimination methods to find the solution of linear algebraic system we need to do row operations.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \vdots \\ b_n \end{Bmatrix}$$

$$\text{i.e.,} \quad A \quad x \;=\; b$$

- Scaling: Any row can be multiplied by a constant. This is not going to change the solution.
- Pivoting: We can interchange the order of rows as per our convenience.
- Elimination: We can replace any row (i.e. a equation) by a weighted linear combination of that row with another row. This may yield some zeroes in that row. This is elimination.
- The row operation are not going to change the solutions.

- Q. So why do we require to do row operations?
- To prevent division by zero.
- To avoid round-off.
- To implement systematic elimination.
- Consider the following linear system example:

$$\begin{pmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}$$

- How do you use to solve such a system:

$$\begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ -20 & 40 & -20 & \vdots & 20 \\ -20 & -20 & 130 & \vdots & 20 \end{bmatrix} \rightarrow \text{The Augmented matrix}$$

$R_2 = R_1 + 4a_{21}$

$R_3 = R_1 + 4a_{31}$

$$\Rightarrow \begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ 0 & 140 & -100 & \vdots & 100 \\ 0 & -100 & 500 & \vdots & 100 \end{bmatrix}$$

$R_3 = 5R_2 + 7R_3$

$$\Rightarrow \begin{bmatrix} 80 & -20 & -20 & \vdots & 20 \\ 0 & 140 & -100 & \vdots & 100 \\ 0 & 0 & 3000 & \vdots & 1200 \end{bmatrix}$$

Now $3000\, x_3 = 1200$

$\therefore x_3 = 0.40$

- Back substituting,

$140\ x_2 - 100\ x_3 = 100$

$\Rightarrow x_2 = 1.0$

$80\ x_1 - 20 \times 1.0 - 20 \times 0.40 = 20$

$\Rightarrow x_1 = 0.60$

$\Rightarrow$ this is a simple elimination method.

$\Rightarrow$ In this process you were actually performing some row operations. You were not knowing them in school days.

- Q. Why do you require scaling?
- As seen in example, we were able to multiply some rows with scalar values. This helped in subsequent elimination
- Q. Why do you require pivoting?
- In such linear systems the elements in major diagonal of the matrix is given a_{ii} *where* $i = 1, 2, 3, \ldots, n$.
- If any $a_{ii} = 0$, then you will face difficulty in the above simple elimination method.

- To avoid that we can do
    - Interchanging of rows (equations)
    - Interchanging of columns (variables)
- This is called pivoting.
- If both rows and columns interchanged, it’s full pivoting else partial pivoting.
- Advantage:
    - We can avoid zero point elements
    - Reduce round-off errors

- Consider the system $$\begin{pmatrix} 0 & 2 & 1 \\ 4 & 1 & -1 \\ -2 & 3 & -3 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -3 \\ 5 \end{Bmatrix}$$
- We can $a_{11} = 0$, the largest element in first column is in row 2. Interchange row 1 and row 2

$$\begin{pmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 2 & 1 & \vdots & 5 \\ -2 & 3 & -3 & \vdots & 5 \end{pmatrix} \quad R_3 = R_1 + 2R_3$$

$$\Rightarrow \begin{pmatrix} 4 & 1 & -1 & \vdots & -3 \\ 0 & 2 & 1 & \vdots & 5 \\ 0 & 7 & -7 & \vdots & 7 \end{pmatrix}$$

- By scaling we can reduce round-off errors.

Gauss Elimination Method

- To solve a linear system [A]{x}={b}, we have to do row operations:
  - Scaling
  - Pivoting
  - Elimination
- While discussing about scaling we saw the example problem.

$$\begin{pmatrix} 3 & 2 & 105 \\ 2 & -3 & 103 \\ 1 & 1 & 3 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 104 \\ 98 \\ 3 \end{Bmatrix}$$

- If the computer program has restriction of three significant digits, then we saw that if we do direct elimination, we are getting erroneous results $x_1 = -0.844$, $x_2 = 0.924$, and $x_3 = 0.997$.
- The errors can be reduced by first doing scaling on the equation and determine its position in the system (i.e. pivoting).
- In direct elimination, we wanted the elements below pivot element as zero
  - ✓ If we scale the numbers, we can pivot the appropriate equation.

- ✓ e.g. for $a_{11} = 3$, we want $a_{21} = a_{31} = 0$.
- ✓ Check the relative values.

  First column values $\begin{Bmatrix} 3 & 2 & 1 \end{Bmatrix}^T$

  w.r.t. the largest values in their equation.

$$\begin{Bmatrix} 3/105 \\ 2/103 \\ 1/3 \end{Bmatrix} = \begin{Bmatrix} 0.0288 \\ 0.0194 \\ \boxed{0.333} \end{Bmatrix}$$

- ✓ This shows that the last row is having the largest scaled values. Therefore it will be appropriate if we pivot this element.

- Therefore pivoting is done by interchanging Row 1 and Row 3.

i.e.,
$$\begin{pmatrix} 1 & 1 & 3 & \vdots & 3 \\ 2 & -3 & 103 & \vdots & 98 \\ 3 & 2 & 105 & \vdots & 104 \end{pmatrix}$$

- Now do row operations – elimination

$$R_2 = R_2 - (a_{21}/a_{11})R_1 = R_2 - 2R_1$$
$$R_3 = R_3 - (a_{31}/a_{11})R_1 = R_3 - 3R_1 \Rightarrow \begin{pmatrix} 1 & 1 & 3 & \vdots & 3 \\ 0 & -5 & 97 & \vdots & 92 \\ 0 & -1 & 96 & \vdots & 95 \end{pmatrix}$$

- Before doing second elimination, i.e., making $a_{32} = 0$, another round of scaling is done to determine pivoting.

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 & \vdots & 3 \\ 0 & -5 & 97 & \vdots & 92 \\ 0 & -1 & 96 & \vdots & 95 \end{pmatrix} R_3 = R_3 - (a_{32} / a_{22})R_2 = R_3 - 0.2R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 & \vdots & 3 \\ 0 & -5 & 97 & \vdots & 92 \\ 0 & 0 & 76.6 & \vdots & 76.6 \end{pmatrix}$$

$$\Rightarrow x_3 = 1.0,\ x_2 = -1.0,\ x_3 = 1.0$$

- Gauss Elimination Method in a Nutshell
- You know that the method is used to solve a linear system $\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{b}$
- Using systematic elimination the above system is converted to $\underset{n\times n}{U}\ \underset{n\times 1}{x} = \underset{n\times 1}{y}$
- [*U*] -> upper triangular matrix

  -> using backsubstitution the solutions x_1, x_2, x_3 are found.

- <u>The Algorithm</u>

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \vdots \\ b_n \end{Bmatrix}$$

- Step 1

○ reduce the elements of first column to zero, except the pivot element

○ The pivot element is a_{11}. If $a_{11} = 0$, do pivoting $a_{r1} = \max_{2 \le i \le n} |a_{i1}|$

○ Identify multiplication factor for each row.

- The multiplying factors are

$$l_{21} = a_{21}/a_{11}, l_{31} = a_{31}/a_{11}, ..., l_{i1} = a_{i1}/a_{11}, ... l_{n1} = a_{n1}/a_{11}$$

- The first column of matrix *A* is $\begin{Bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{Bmatrix}$ except a11, all other quantities have to be zero.
- Now we need to multiply the first row by t he multiplying factor (l_{i1}) and deduct it from the corresponding row (*i*).
    - Due to this changes occur in vector{*b*} also.

- For our convenience the step number is given or bracketed superscript and we can decide this calculation as:

$$a_{ij}^{(1)} = a_{ij} - l_{i1}a_{1j}$$

$$b_i^{(1)} = b_i - l_{i1}b_1 \quad ; i = 2,3,4,\ldots,n \text{ and } j = 1,2,3,4,\ldots,n$$

- After the first steps you have:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2j}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & \cdots & a_{3j}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nj}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ \vdots \\ b_n^{(1)} \end{Bmatrix}$$

- Step 2
  - Adjust similar procedure as in step 1.
  - However now the pivot element is $a_{22}^{(1)}$.
  - We need to eliminate elements in the second column below $a_{22}^{(1)}$.
  - If $a_{22}^{(1)} = 0$, then do pivoting $a_{r2} = \max_{3 \le i \le n} \left| a_{i2}^{(1)} \right|$
  - Compute multiplying factors for each row below row 2.

i.e., $l_{32} = a_{32}^{(1)} / a_{22}^{(1)}, l_{42} = a_{42}^{(1)} / a_{22}^{(1)}, \ldots l_{n2} = a_{n2}^{(1)} / a_{22}^{(1)}$

i.e., in general $l_{i2} = a_{i2}^{(1)} / a_{22}^{(1)};\ i = 3, 4, 5, \ldots, n$

- Eliminate all elements below $a_{22}^{(1)}$ as zeroes.

For that do

$$\left.\begin{aligned} a_{ij}^{(2)} &= a_{ij}^{(1)} - l_{i2} a_{2j}^{(1)} \\ b_i^{(2)} &= b_i^{(1)} - l_{i2} b_2^{(1)} \end{aligned}\right\} i = 3,4,5,\ldots,n \text{ and } j = 2,3,4,5,\ldots,n$$