CE 601: Numerical Methods

Lecture 29

Partial Differential Equations

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- We already discussed how to use finite-difference methods to solve boundary-value ODEs.
- A general linear second-order BV-ODE is of the form $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$
- You have also seen

  - Dirichlet B.C.
  - Neumann B.C.
  - Mixed B.C.

- However in most real world problems, the governing equation developed through fundamental principles may be in partial differential forms.
- You require partial differential equations.
- What is a PDE?
- The dependent variable for a problem depends on more than one independent variable and the governing equations consist of partial derivatives.

Some examples of PDEs are:

- $$\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}=0;$$ Laplce equation

- $$\frac{\partial C}{\partial t}=\alpha\frac{\partial^2 C}{\partial x^2};$$ (Diffusion equation)

- $$\frac{\partial^2 C}{\partial t^2}=m\frac{\partial^2 C}{\partial x^2};$$ (Wave equation)

- $$\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}=F(x,y,z);$$ (Poisson's equation)

- <u>Classification of PDEs</u>:
- Here onwards our dependent variable will be f. Independent variables are mostly space (x, y, z) and time t.
- A general quasi-linear second order non-homogeneous PDE in two independent variables is given as

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial y^2} + C\frac{\partial^2 f}{\partial z^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff = G$$

- We can now proceed to define the discriminant of the coefficients as $B^2 - 4AC$

- If $B^2 - 4AC < 0;$ (Elliptic PDE)

  $B^2 - 4AC = 0;$ (Parabolic PDE)

  $B^2 - 4AC > 0;$ (Hyperbolic PDE)

- Example: $\text{Laplace equation}: \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0;$

  $\text{Here, } B^2 - 4AC = -4 < 0$

  Hence, elliptic PDE.

- Diffusion equation: $\dfrac{\partial f}{\partial t} = \alpha \dfrac{\partial^2 f}{\partial x^2}$;

Here, $A = \alpha, B = 0, C = 0 \therefore B^2 - 4AC = 0$

Hence, parabolic PDE.

- Wave equation: $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$;

Here, $A = 1, B = 0, C = -c^2 \therefore B^2 - 4AC = 4c^2 > 0$

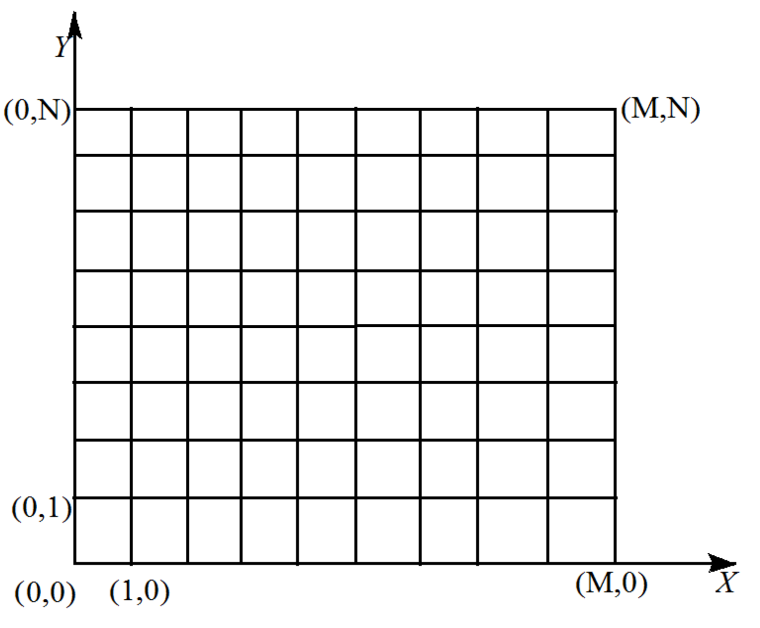
Hence, hyperbolic PDE.

- Elliptic PDE:
- As usual, depending on the type of PDE, we need to specify the initial and boundary conditions. We will utilize the finite difference methods for solving the elliptic PDEs in this discussion
- FDM for Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0;$$

holds true in case of the x-y plane.

- To find the solution of the above equation, we can employ the finite difference method.
- The x-y plane is discretized into finite-difference rectangle.



- Assign grid node numbers

$$(i,\, j) = (0,0), (1,0), (0,1), (1,1), \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{M}, \mathrm{N}).$$

- At any grid node (*i,j*), $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(i,j)} + \left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(i,j)} = 0$

or, $$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta x^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta y^2} = 0$$

The second order centered space approximation has been used for the second derivative.

- Let us define $\beta = \dfrac{\Delta x}{\Delta y}$. This is called the grid aspect ratio.
- The FDE becomes

$$f_{i-1,j} + \beta^2 f_{i,j-1} - 2\;\; 1+\beta^2 \;\;\; f_{i,j} + f_{i,j+1} + f_{i+1,j} = 0$$

- In case $\beta = 1,\ (\Delta x = \Delta y)$, the FDE becomes

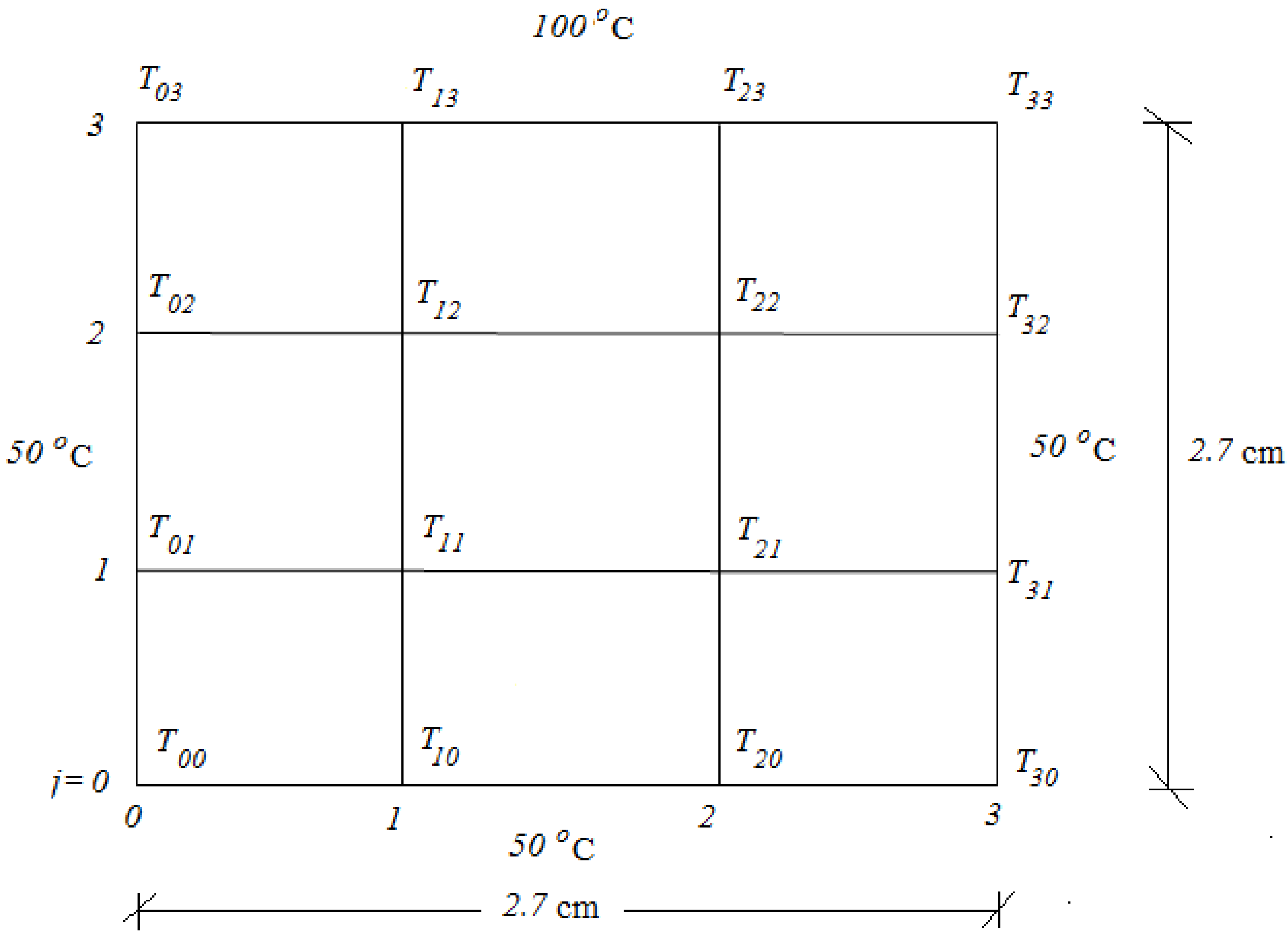
$$f_{i-1,j} + f_{i,j-1} - 4.f_{i,j} + f_{i,j+1} + f_{i+1,j} = 0$$

- In the above equation, information of five grid points are involved. Hence, they are called five point schemes.

- The FDE is applied to all grid nodes and the subsequent system of algebraic equations are formulated.
- **Example**:

  A vertical steel plate of dimensions 2.7 cm X 2.7 cm and negligible thickness is in steady state conditions. On the top edge, the temperature is 100^{o}C and on the bottom the temperature is fixed at 50^{o}C. The temperature on the left and right edges are 50^{o}C. Solve to obtain heat distribution.

- **Solution**: The vertical plate is discretised as below:



The governing equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

$T(x = 2.7, y) = 50^o\text{C}$; $T(x = 0, y) = 50^o\text{C}$;

$T(x, y = 2.7) = 100^o\text{C}$; $T(x, y = 0) = 50^o\text{C}$.

$\Delta x = \Delta y = 0.9$ cm.

We know the temperatures,

$T_{00} = 50^o\text{C},\ T_{01} = 50^o\text{C},\ T_{02} = 50^o\text{C},\ T_{03} = 50^o\text{C},$

$T_{10} = 50^o\text{C},\ T_{20} = 50^o\text{C},\ T_{30} = 50^o\text{C},\ T_{31} = 50^o\text{C},$

$T_{32} = 50^o\text{C},\ T_{03} = 100^o\text{C},\ T_{13} = 100^o\text{C},\ T_{23} = 100^o\text{C},\ T_{33} = 100^o\text{C}.$

We need to find temperatures $T_{11}, T_{12}, T_{21}, T_{22}$ (four unknowns).

Apply FDE <u>at node (1,1)</u>,

$T_{01} + T_{10} - 4T_{11} + T_{12} + T_{21} = 0$

i.e. $50 + 50 - 4T_{11} + T_{12} + T_{21} = 0$

or, $-4T_{11} + T_{12} + T_{21} = -100 \quad \rightarrow (1)$

Node or grid point $(1,2)$,

$$T_{02}+T_{11}-4T_{12}+T_{13}+T_{22}=0$$

i.e. $50+T_{11}-4T_{12}+100+T_{22}=0$

or, $T_{11}-4T_{12}+T_{22}=-150 \quad \rightarrow (2)$

Node $(2,1)$,

$$T_{11}+T_{20}-4T_{21}+T_{22}+T_{31}=0$$

i.e. $T_{11}+50-4T_{21}+T_{22}+50=0$

or, $T_{11}-4T_{21}+T_{22}=-100 \quad \rightarrow (3)$

Node $(2,2)$,

$$T_{12}+T_{21}-4T_{22}+T_{23}+T_{32}=0$$

i.e. $T_{12}+T_{21}-4T_{22}=-150 \quad \rightarrow (4)$

From (1), (2), (3) & (4), we have,

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 0 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{Bmatrix} = \begin{Bmatrix} -100 \\ -150 \\ -100 \\ -150 \end{Bmatrix}$$

Solve this system of linear equations.

- As discussed for ODEs we should note about

    - Consistency,
    - Order,
    - Convergence.