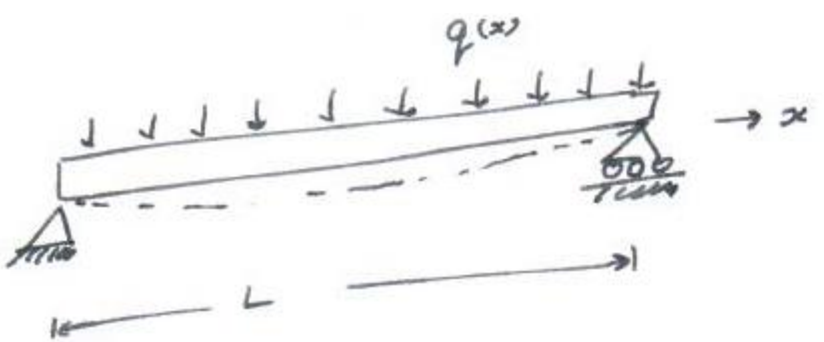
CE 601: Numerical Methods

Lecture 28

Boundary-Value Ordinary Differential Equations

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

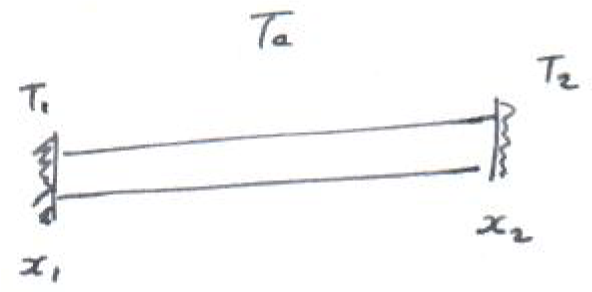
- As suggested, based on the auxiliary conditions we can define ODE as:
  - IV-ODE
  - BV-ODE
- Some examples of BV-ODE:
- Beam bending problem



$$EI\frac{d^4y}{dx^4} = q(x); y(0)=0, y(L)=0, y''(0)=0, y''(L)=0$$

$y \rightarrow$ deflection (dependent variable)

- Heat diffusion in a steel rod:



$$\frac{d^2T}{dx^2} - \beta^2 T = -\beta^2 T_a$$

$$T(x_1) = T_1$$
$$T(x_2) = T_2$$

- The dependent variable varying w.r.t space generally leads to boundary-value ODEs.

- Boundary value ODEs are often found in equilibrium problems and in closed domains.
- Following types of BV-ODEs are generally encountered:
  – Single Boundary Value ODE
  – System of Boundary Value ODEs
  – Linear and Non-linear ODEs
  – Homogeneous and Non-homogeneous ODEs

- A general second-order non-linear BV-ODE is:

$$\frac{d^2y}{dx^2} + P \;\; x, y \;\; \frac{dy}{dx} + Q \;\; x, y \;\; y = F(x); \qquad y(x_1) = y_1$$

$$y(x_2) = y_2$$

- A general second order linear BV-ODE is:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x); \qquad y(x_1) = y_1$$

$$y(x_2) = y_2$$

- Solution domain of the above equations is

$$x_1 \le x \le x_2$$

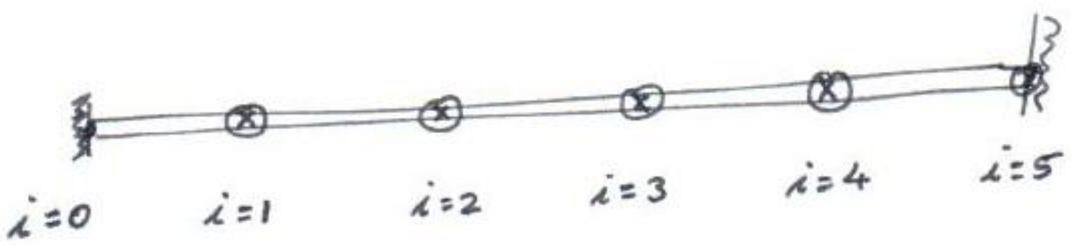
- Boundary conditions for BV-ODEs are:
  - Dirichlet Boundary condition $\text{e.g.} \Rightarrow y(x_1) = c_1$
  - Neumann Boundary condition $\left(\text{e.g.} \Rightarrow \left.\frac{dy}{dx}\right|_{x_1} = c_2\right)$
  - Mixed Boundary condition $\left(\text{e.g.} \Rightarrow ay(\mathrm{x}_1) + b\left.\frac{dy}{dx}\right|_{x_1} = c_3\right)$

The Equilibrium Finite-Difference Method

- We will use the equilibrium finite difference method to solve the boundary-value ODEs.
  - Discretize the entire continuous spatial domain into smaller discrete grid points.
  - Apply the ODE at each grid point.
  - Approximate the derivatives at any grid point using finite difference formulas.
  - Obtain the corresponding finite-difference algebraic equations.
  - Solve them

- Consider the following 1-D second-order linear BV-ODE:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x); \quad y(x_0) = y_0, \quad y(x_L) = y_L$$



- The domain has been discretized.

- At any general node i, we have

$$\left.\frac{d^2y}{dx^2}\right|_i + P(x_i)\left.\frac{dy}{dx}\right|_i + Q(x_i)y_i = F(x_i) \qquad \rightarrow (1)$$

- Now, $\left.\frac{d^2y}{dx^2}\right|_i \simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \quad ; \qquad O(\Delta x^2)$

$$\left.\frac{dy}{dx}\right|_i \simeq \frac{y_{i+1} - y_{i-1}}{2\Delta x} \quad ; \qquad O(\Delta x^2)$$

- Substituting in (1), we have

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}\right) + P_i\left(\frac{y_{i+1} - y_{i-1}}{2\Delta x}\right) + Q_i y_i = F_i$$

- Rearranging the terms, we have

$$y_{i-1}\left(\frac{1}{\Delta x^2}-\frac{P_i}{2\Delta x}\right)+y_i\left(-\frac{2}{\Delta x^2}+Q_i\right)+y_{i+1}\left(\frac{1}{\Delta x^2}+\frac{P_i}{2\Delta x}\right)=F_i$$

or

$$\boxed{\left(1-\frac{\Delta x}{2}P_i\right)y_{i-1}+\;\; -2+\Delta x^2 Q_i \;\; y_i+\left(1+\frac{\Delta x}{2}P_i\right)y_{i+1}=F_i\Delta x^2} \qquad \rightarrow (2)$$

Equation (2) is the finite-difference equation for grid node *'i'*.

This equation is applied to each grid node *'i'* to generate a system of algebraic equations.

- **Example**: Solve the heat transfer BV-ODE for a rod of length 1.0 cm. The governing ODE is:

$$\frac{d^2T}{dx^2} - \beta^2 T = -\beta^2 T_a$$

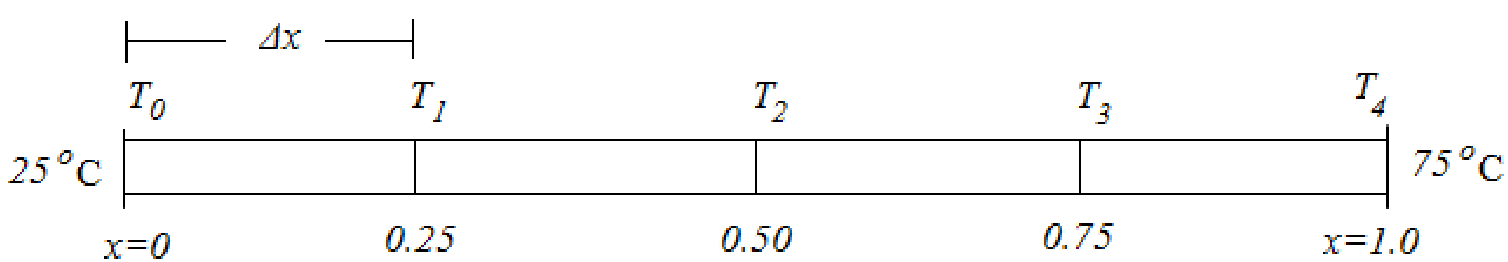
where $\beta^2 = 16.0 \text{ cm}^{-2}$ (heat diffusivity)

$T_a = 20^\circ \text{C}$ (ambient temperature)

The B.Cs $T(0.0) = 25^\circ \text{C}$

$T(1.0) = 75^\circ \text{C}$

- **Solution**: The rod is of length = 1.0 centimeter (cm). Let us discretise the length as follows:



$\Delta x = 0.25$ cm. As per the given data $T_0 = 25^0$C and $T_4 = 75^0$C. T_1, T_2, T_3 are unknowns.

For any node: $\left.\frac{d^2T}{dx^2}\right|_i - 16T_i = -16\times 20 = -320$

i.e. $\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - 16T_i = -320$

i.e. $T_{i+1} - 2T_i + T_{i-1} - T_i = -20$

i.e. $\boxed{T_{i+1} - 3T_i + T_{i-1} = -20} \quad \to (1)$

Eq.(1) is applied at unknown nodes 1,2,3.

At $i = 1$,

$T_2 - 3T_1 + T_0 = -20$

$\Rightarrow T_2 - 3T_1 + 25 = -20$

$\Rightarrow -3T_1 + T_2 = -45$

At $i = 2$,

$T_1 - 3T_2 + T_3 = -20$

At $i = 3$,

$T_2 - 3T_3 + T_4 = -20$

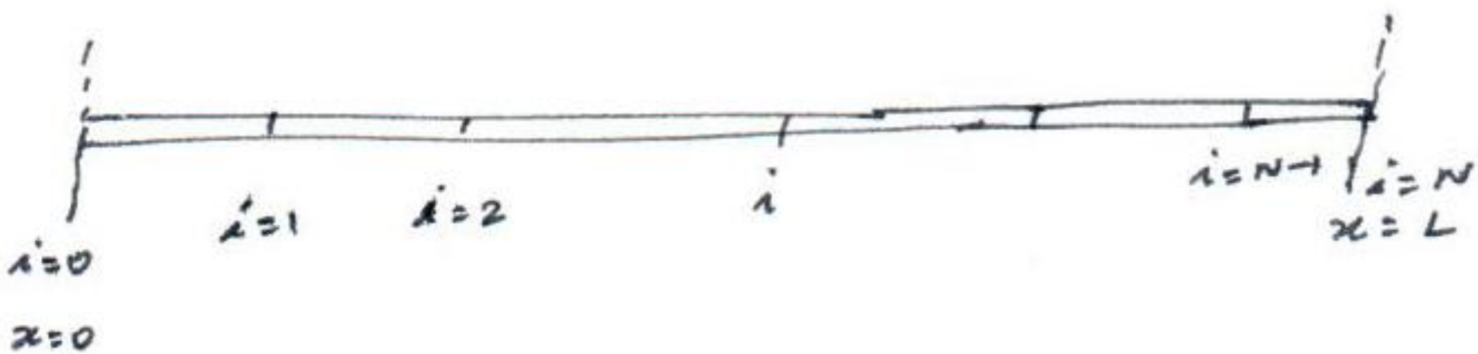
$\Rightarrow T_2 - 3T_1 = -95$

i.e. $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -45 \\ -20 \\ -95 \end{Bmatrix}$

$\to$ Solve this system of linear equations to get T_1, T_2, T_3.

- <u>To solve problems with Neumann B.C.s</u>

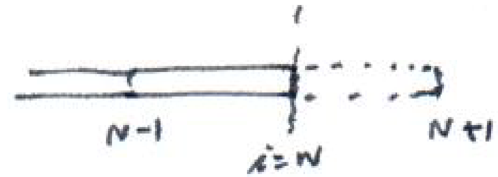
$$\frac{d^2y}{dx^2}+P\;\;x\;\;\frac{dy}{dx}+Q\;\;x\;\;y=F(x);\;y(x_0)=y_0,\;\;\left.\frac{dy}{dx}\right|_{x=L}=m$$



- The FDE (2) is applied in each node *‘i’*:

$$\left(1-\frac{\Delta x}{2}P_i\right)y_{i-1} + \;\; -2+\Delta x^2 Q_i \;\; y_i + \left(1+\frac{\Delta x}{2}P_i\right)y_{i+1} = F_i \Delta x^2$$

- At $i = N$, Neumann B.C. is given.



- Consider an imaginary node *N+1,*

i.e. not physically present and is used only for computational purpose.

$$\left.\frac{dy}{dx}\right|_{x_N} = m \quad \Rightarrow \quad \frac{y_{N+1}-y_{N-1}}{2\Delta x} = m \quad \Rightarrow y_{N+1} = y_{N-1} + 2m\Delta x$$

- Substitute the previous expression for y_{N+1} at node $i = N$ (FDE)

$$\text{i.e. } \left(1-\frac{\Delta x}{2}P_N\right)y_{N-1} + \; -2+\Delta x^2 Q_N \;\; y_N + \left(1+\frac{\Delta x}{2}P_N\right)y_{N+1} = F_N \Delta x^2$$

$$\text{i.e. } \left\{\left(1-\frac{\Delta x}{2}P_N\right)+\left(1+\frac{\Delta x}{2}P_N\right)\right\}y_{N-1} + \; -2+\Delta x^2 Q_N \;\; y_N = F_N \Delta x^2 - \left(1+\frac{\Delta x}{2}P_N\right)2m.\Delta x$$

$$\text{or, } \boxed{2y_{N-1} + \; -2+\Delta x^2 Q_N \;\; y_N = F_N \Delta x^2 - 2m.\Delta x - m.\Delta x^2 P_N} \rightarrow (3)$$

- The FDE (3) is to be applied at the boundary node, where Neumann B.C. is given.

In the above-mentioned example

if $\left.\frac{dT}{dx}\right|_{x=0} = 0$ and $T(x_L) = 75^\circ \text{C}$ are the boundary conditions,

then the FDE is to be applied in nodes $i = 0, 1, 2, 3.$

For $i = 0$, $\left.\frac{dT}{dx}\right|_{x=0} = 0 \quad \Rightarrow \frac{T_1 - T_{-1}}{2\Delta x} = 0 \quad \Rightarrow T_1 = T_{-1}$

$\therefore$ At $i = 0$, $\quad T_{-1} - 3T_0 + T_1 + 0 = -20$

$$\Rightarrow -3T_0 + 2T_1 + 0 + 0 = -20$$

At $i = 1$, $\quad T_0 - 3T_1 + T_2 + 0 = -20$

At $i = 2$, $\quad 0 + T_1 - 3T_2 + T_3 = -20$

At $i = 3$, $\quad 0 + 0 + T_2 - 3T_3 = -95$

$\therefore$ System of linear equations: $\begin{pmatrix} -3 & 2 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ -20 \\ -20 \\ -95 \end{Bmatrix}$

- <u>To solve Mixed B.C. problems</u>:
- If the mixed B.C. is given at one-end for the general linear second order (1-D) BV-ODE

$$Ay(\mathrm{x}_N) + B\left.\frac{dy}{dx}\right|_{x_N} = p$$

- Then at node *i=N*, the FDE

$$Ay_N + B\left(\frac{y_{N+1} - y_{N-1}}{2\Delta x}\right) = p$$

$$\text{or, } B \;\; y_{N+1} - y_{N-1} \;\; = 2p\Delta x - 2A\Delta x y_N$$

$$\text{or, } y_{N+1} = y_{N-1} + \frac{1}{B} \;\; 2p\Delta x - 2A\Delta x y_N$$

Substitute this expression of y_{N+1} in FDE (2).