CE 601: Numerical Methods

Lecture 27

Multi-Point Methods

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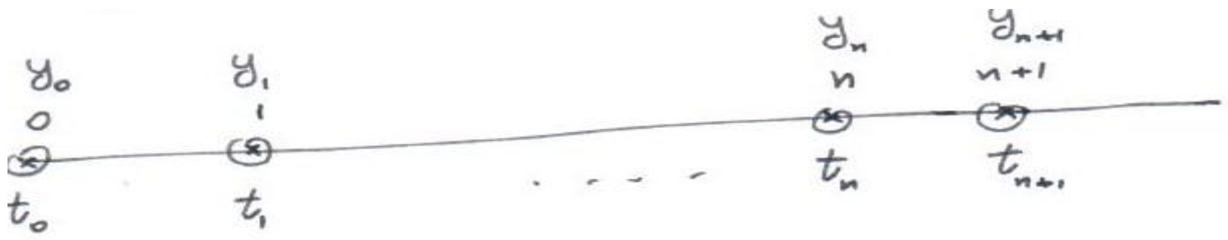
- <u>Multi Point Methods:</u>
- The Euler's methods and Runge-Kutta methods that were discussed till now for solving IV-ODEs were single step (or single point) methods.
- In the time scale, only the information of y_n was considered for obtaining y_{n+1}



- If the information of more than one previous instant is used, then the method is called a multi-point method.
- Principle of Multi-Point methods:

$$\frac{dy}{dt} = f(t, y)$$

- We can approximately represent the above equation as

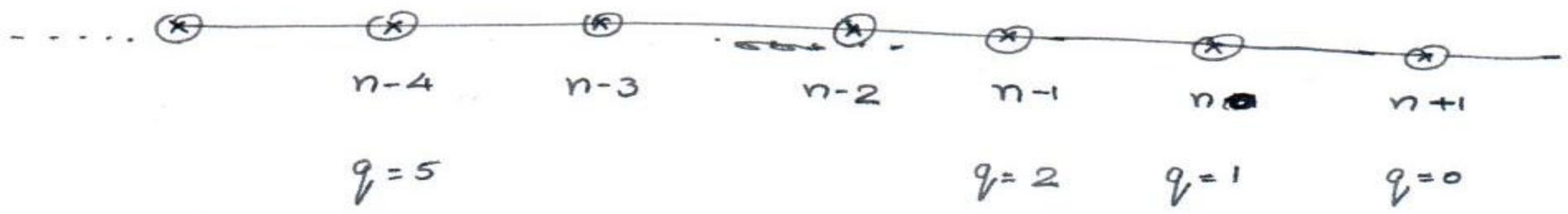
$$dy = f(t, y(t))dt = F(t)dt$$

$$\int dy = \int F(t)dt$$

- $F(t)$ at different instants is approximated by Newton's backward difference polynomials.

$$\int dy = \int P_k(t)dt$$

- So, for the discrete time domain, we have



$$\int_{y_{n+1-q}}^{y_{n+1}} dy = \int_{t_{n+1-q}}^{t_{n+1}} P_k(t)dt$$

- If $P_k(t)$ is obtained with base point time t_n , the resulting expression will be explicit multi-step equation.
- If $P_k(t)$ is obtained with base point time t_{n+1} , then the resulting expression will be implicit multi-step equation.

- If $q=1$, we have $\int_{y_n}^{y_{n+1}} dy = \int_{t_n}^{t_{n+1}} P_k(t)dt = \Delta y \quad \rightarrow (1)$
- This integration (1) is called Adams’ finite-difference equations.
- Explicit ones are called Adams-Bashforth FDEs.
- Implicit ones are called Adams-Moulton FDEs.

- <u>Fourth Order Adams Bashforth Moulton Method:</u>
- It has already been seen that

$$\int_{y_n}^{y_{n+1}} dy = \int_{t_n}^{t_{n+1}} P_k(t)dt = \Delta y$$

- In this case, a third degree polynomial is integrated to obtain the fourth degree polynomial. So,

$$\Delta y = \int_{t_n}^{t_{n+1}} P_3(t)\ _n\ dt$$

- The explicit solution has been considered first.

$$P_3(\mathrm{s}) = f_n + s.\nabla f_n + \ s(s+1)/2! \ \nabla^2 f_n + \ s(s+1)(\mathrm{s}+2)/3! \ \nabla^3 f_n, \ O \ \ \Delta t^4$$

$$s = \frac{t - t_n}{\Delta t} \Rightarrow ds = \frac{dt}{\Delta t}$$

when

$$t = t_n, s = 0$$

$$t = t_{n+1}, s = 1$$

$$\text{So, } \Delta y = y_{n+1} - y_n = \Delta t \int_0^1 P_3(s) ds$$

$$\text{i.e. } y_{n+1} = y_n + \Delta t \int_0^1 \left[f_n + s.\nabla f_n + \ s(s+1)/2! \ \nabla^2 f_n + \ s(s+1)(\mathrm{s}+2)/3! \ \nabla^3 f_n \right] ds$$

$$\text{i.e. } y_{n+1} = y_n + \Delta t \left[f_n + \frac{1}{2} \nabla f_n + \frac{5}{12} \nabla^2 f_n + \frac{3}{8} \nabla^3 f_n \right]$$

- Backward difference table is as follows:

t	f	∇f	$\nabla^2 f$	$\nabla^3 f$
t_{n-3}	f_{n-3}			
		$f_{n-2} - f_{n-3}$		
t_{n-2}	f_{n-2}		$f_{n-1} - 2.f_{n-2} + f_{n-3}$	
		$f_{n-1} - f_{n-2}$		$f_n - 3.f_{n-1} + 3.f_{n-2} + f_{n-3}$
t_{n-1}	f_{n-1}		$f_n - 2.f_{n-1} + f_{n-2}$	
		$f_n - f_{n-1}$		$f_{n+1} - 3.f_n + 3.f_{n-1} + f_{n-2}$
t_n	f_n		$f_{n+1} - 2.f_n + f_{n-1}$	
		$f_{n+1} - f_n$		
t_{n+1}	f_{n+1}			

- So, substituting all the relevant values in the equation, we have

$$y_{n+1} = y_n + \Delta t\left[f_n + \frac{1}{2}\;\; f_n - f_{n-1} \;\; + \frac{5}{12}\;\; f_n - 2.f_{n-1} + f_{n-2} \;\; + \frac{3}{8}\;\; f_n - 3.f_{n-1} + 3.f_{n-2} - f_{n-3} \;\; \right]$$

Or,

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{24}\;\; 55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}}$$

This is called Adams-Bashforth fourth order explicit FDE.

- Similarly, in case we integrate using implicit approach,

$$\Delta y = \int_{t_n}^{t_{n+1}} P_3(t) \;_{n+1}\, dt = \int_{-1}^{0} P(s)ds + \text{error}$$

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{24}\; 9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + 9\,f_{n-2}}$$

- This is the fourth order Adams-Moulton implicit FDE.
- As it is observed difficulty in evaluating *'f'* in implicit condition, so Adams-Moulton FDE can be simplified by predictor-corrector approach (known as 4th order Adams-Bashforth-Moulton predictor corrector method):

$$\boxed{\begin{aligned} y_{n+1}{}^{P} &= y_n + \frac{\Delta t}{24}\; 55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3} \\ y_{n+1}{}^{C} &= y_n + \frac{\Delta t}{24}\left[9\,f_{n+1}{}^{P} + 19\,f_n - 5\,f_{n-1} + 9\,f_{n-2}\right] \end{aligned}}$$

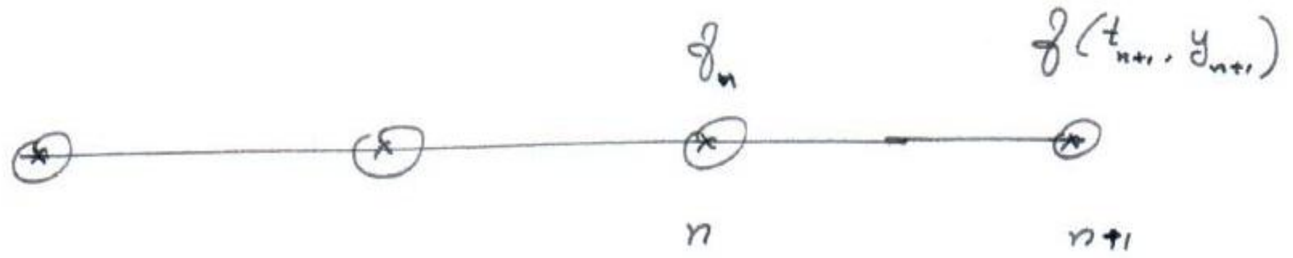
- <u>To solve IV-ODE's having Non-Linear Derivative Functions</u>
- For the IV-ODE $\frac{dy}{dt} = f(t, y); y(t_0) = y_0$

  it has been seen that explicit and implicit methods can be employed. However, when $f(t, y)$ is non-linear in y , then employing the implicit methods become quite tedious and difficult.

- Let us consider the Euler implicit method

$$y_{n+1} = y_n + \Delta t . f\left(t_{n+1}, y_{n+1}\right)$$

- Here, $f\left(t_{n+1}, y_{n+1}\right)$ is non-linear. So, in the time scale



$y_n + \Delta t . f\left(t_{n+1}, y_{n+1}\right)$ is unknown in y_{n+1}

Let us write it as $G(y_{n+1})$

- Hence, we will need to solve the equation

$$y_{n+1} - G(y_{n+1}) = 0$$

- We need to find out the value of y_{n+1} using iterative procedures. Let us define

$$F(y_{n+1}) = y_{n+1} - G(y_{n+1})$$

- Newton Raphson method suggests that

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \frac{F\ \ y_{n+1}^{(k)}}{F'\ \ y_{n+1}^{(k)}}$$

- Modified Newton Raphson method suggests that

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \frac{F\left(y_{n+1}^{(k)}\right)}{F'}$$

- Here, F' is calculated only once.

- Example: Solve

$$\frac{dT}{dt} = -\alpha \ \ T^4 - T_a^4 \ \ ; \qquad T_a = 250K, \ \ T_0 = 2500K, \ \ \alpha = 4 \times 10^{-12}$$

using Euler’s implicit method and apply Newton Raphson method.

- Solution:

$$T_{n+1} = T_n + \frac{\Delta t}{2} \cdot \ \ f_n + f_{n+1}$$

where

$$f = - \ \ 4 \times 10^{-12} \ \ \cdot \ \ T^4 - 250^4$$

- Using $\Delta t = 2.0s$, we have

$$T_1 = T_0 + \left(\frac{2.0}{2}\right) \cdot \quad -4.0 \times 10^{-12} \quad 2500^4 - 250^4 + T_1^4 - 250^4$$

$$T_1 = \quad 2343.78 - 4 \times 10^{-12} \times T_1^4 \quad \Rightarrow G \quad T_1$$

$$F \quad T_1 \quad = T_1 - \quad 2343.78 - 4 \times 10^{-12} \times T_1^4$$

$$F' \quad T_1 \quad = 1 - 1.6 \times 10^{-11} \times T_1^3$$

- Initial guess is taken as $T_1^{(0)} = 2500$

$$T_1^{(1)} = T_1^{(0)} - \frac{F \quad T_1^{(0)}}{F' \quad T_1^{(0)}}$$

$$T_1^{(1)} = 2500 - \frac{312.4700}{0.7500} = 2083.373$$

$$T_1^{(2)} = T_1^{(1)} - \frac{F\ \ T_1^{(1)}}{F'\ \ T_1^{(1)}}$$

$$T_1^{(2)} = 2083.373 - \frac{-185.049}{0.8553} = 2299.725$$

$$T_1^{(3)} = 2299.725 - \frac{67.828}{0.8054} = 2215.508$$

$$T_1^{(4)} = 2215.508 - \frac{-31.899}{0.826} = 2254.127$$

$$T_1^{(5)} = 2254.127 - \frac{13.616}{0.8167} = 2237.454$$

- Eventually, after several iterations, we arrive at the answer as $T_1 = 2242.605K$.