CE 601: Numerical Methods

Lecture 26

Runge-Kutta Methods-II

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Example: Solve the IV-ODE $\dfrac{dT}{dt} = -\alpha\left(T^4 - T_a{}^4\right)$ where $T_a = 250K, T(t_0) = 2500K,$ $\alpha = 4\times10^{-12}$. Use $\Delta t = 2$ seconds.

Soln.

$$\Delta \text{t} = 2 \text{ s}, T_0 = 2500K, T_a = 250K. \begin{cases} \text{The time line is} \\ \quad \text{T}_0 \qquad\qquad \text{T}_1 \qquad\qquad \text{T}_2 \qquad\qquad \text{T}_3 \\ \quad \otimes \text{———} \otimes \text{———} \otimes \text{———} \otimes \cdots \\ \text{t}_0 = 0 \qquad \text{t}_1 = 2 \qquad \text{t}_2 = 4 \qquad \text{t}_3 = 8 \end{cases}$$

$$f(t,T) = \left(-4\times10^{-12}\right)\times\left(T^4 - 250^4\right).$$

The 4^{th} order R-K method:

$$T_{n+1} = T_n + \frac{1}{6}\left(\Delta T_1 + 2\Delta T_2 + 2\Delta T_3 + \Delta T_4\right)$$

where $\Delta T_1 = \Delta t \times f(t_n, T_n)$, $\Delta T_2 = \Delta t \times f\left(t_n + \dfrac{\Delta t}{2}, T_n + \dfrac{\Delta T_1}{2}\right)$,

$$\Delta T_3 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, T_n + \frac{\Delta T_2}{2}\right), \ \Delta T_4 = \Delta t \times f\left(t_n + \Delta t, T_n + \Delta T_3\right)$$

Beginning with $n = 0, T_0 = 2500$ K.

$$\therefore T_1 = T_0 + \frac{1}{6}\left(\Delta T_1 + 2\Delta T_2 + 2\Delta T_3 + \Delta T_4\right)$$

$$\Delta T_1 = \Delta t \times f(t_0, T_0) = 2.0 \times \left(-4 \times 10^{-12}\right) \times \left(2500^4 - 250^4\right) = -156.234.$$

$$\Delta T_2 = 2.0 \times \left(-4 \times 10^{-12}\right) \times \left(\left(2500 - \frac{156.234}{2}\right)^4 - 250^4\right) = -275.203.$$

$$\Delta T_3 = 2.0 \times \left(-4 \times 10^{-12}\right) \times \left(\left(2500 - \frac{275.203}{2}\right)^4 - 250^4\right) = -249.143.$$

$$\Delta T_4 = 2.0 \times \left(-4 \times 10^{-12}\right) \times \left(\left(2500 - 249.143\right)^4 - 250^4\right) = -205.313.$$

$$\therefore T_1 = 2500 + \frac{1}{6} \times \left(-156.234 - 2 \times 275.203 - 2 \times 249.143 - 205.313\right)$$

$$= 2264.96 \text{ K}$$

Similarly evaluate $T_2, T_3, \ldots$

Stability analysis of 4th order R-K method

To perform stability analysis, earlier we suggested that the given non-linear IV-ODE should be converted to the lineat differential form

$$\frac{dy}{dt} + \alpha y = 0;\ y(t_0) = y_0$$

Here you have $f(t, y) = -\alpha y$ and $\alpha \to$ a constant or function of $'t'$ only.

$\therefore$ Note that $f = -\alpha y$

In 4th order R-K method,

$$y_{n+1} = y_n + \frac{1}{6}\left[\Delta y_1 + 2\Delta y_2 + 2\Delta y_3 + \Delta y_4\right]$$

Here $\Delta y_1 = \Delta t \times f_n = \Delta t(-\alpha y) = -(\alpha \Delta t) y_n$

$$\Delta y_2 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_1}{2}\right)$$

$$= \Delta t \times \left[-\alpha\left\{y_n - \frac{(\alpha \Delta t) y_n}{2}\right\}\right]$$

$$= -(\alpha \Delta t) y_n \left[1 - \frac{\alpha \Delta t}{2}\right]$$

$$\Delta y_3 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_2}{2}\right)$$

$$= \Delta t \times \left[-\alpha\left\{y_n + \frac{1}{2}(-\alpha\Delta t y_n)\left(1 - \frac{\alpha\Delta t}{2}\right)\right\}\right]$$

$$= -(\alpha\Delta t) y_n \times \left[1 - \frac{\alpha\Delta t}{2} + \frac{(\alpha\Delta t)^2}{4}\right]$$

$$\Delta y_4 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, y_n + \Delta y_3\right)$$

$$= \Delta t \times \left[-\alpha\left\{y_n + (-\alpha\Delta t y_n)\left(1 - \frac{\alpha\Delta t}{2} + \frac{(\alpha\Delta t)^2}{4}\right)\right\}\right]$$

$$= -(\alpha\Delta t) y_n \times \left[1 - (\alpha\Delta t) + \frac{(\alpha\Delta t)^2}{2} - \frac{(\alpha\Delta t)^3}{4}\right]$$

$\therefore\ y_{n+1}$ can be expressed in single step now using expressions of $\Delta y_1, \Delta y_2, \Delta y_3$ & Δy_4.

We get, $y_{n+1} = y_n\left[1 - (\alpha\Delta t) + \frac{(\alpha\Delta t)^2}{2} - \frac{(\alpha\Delta t)^3}{6} + \frac{(\alpha\Delta t)^4}{24}\right]$.

Here the amplification factor,

$$G = \left[1 - (\alpha \Delta t) + \frac{(\alpha \Delta t)^2}{2} - \frac{(\alpha \Delta t)^3}{6} + \frac{(\alpha \Delta t)^4}{24} \right]$$

We have framed the linear differential equation $\frac{dy}{dt} + \alpha y = 0$, such a way that α is positive.

Now for $|G| \leq 1.0$, we need to have $\alpha \Delta t \leq 2.785$.

You can manually work it out.

For the example problem $\frac{dT}{dt} = -\alpha (T^4 - T_a^4)$;

$T_a = 250$ K and $T(t_0) = 2500$ K, $\alpha = 4 \times 10^{-12} / (\mathrm{K}^3 \mathrm{s})$

If we want to check the stability criteria, we need to linearise the differential equation to the form $\frac{dy}{dt} + \alpha y = F(t)$.

For that, let us write $\alpha = 4 \times 10^{-12}$ as D in the

equation $\dfrac{dT}{dt} = -D\left(T^4 - T_a^4\right)$

i.e. $\dfrac{dT}{dt} = -\left(4 \times 10^{-12}\right)\left(T^4 - 250^4\right)$

Now the general first degree linear IV-ODE is

$\dfrac{dy}{dt} + \alpha y = F(t)$, or $\dfrac{dy}{dt} = -\alpha y + F(t)$.

Here the derivative function $f(t, y)$ can be expressed using

Taylor's series and (t_0, y_0) as base point.

$$f(t, y) = f_0 - \left.\frac{\partial f}{\partial t}\right|_0 t_0 - \left.\frac{\partial f}{\partial y}\right|_0 y_0 + \left.\frac{\partial f}{\partial t}\right|_0 t + \left.\frac{\partial f}{\partial t}\right|_0 y + \cdots \quad \rightarrow (1)$$

In the general linear IV-ODE

$$\frac{dy}{dt} + \alpha y = F(t)$$

We can represent it in general non-linear ODE as

$$\frac{dy}{dt} = -\alpha y + F(t) \quad \left(\frac{dy}{dt} = f(t, y) \right)$$

i.e. $f(t, y) = F(t) - \alpha y \qquad \rightarrow (2)$

Comparing equations (1) and (2),

$$\alpha = -\left.\frac{\partial f}{\partial y}\right|_{t_0, y_0}$$

$$F(t) = f_0 - \left.\frac{\partial f}{\partial t}\right|_0 t_0 - \left.\frac{\partial f}{\partial y}\right|_0 y_0 + \left.\frac{\partial f}{\partial t}\right|_0 t$$

This way the non-linear ODE can be represented in linear fom.

$\therefore$ For $\dfrac{dT}{dt} = -D\left(T^4 - 250^4\right)$

i.e. $\dfrac{dT}{dt} = -\left(4\times10^{-12}\right)\times\left(T^4 - 250^4\right)$ can be converted to $\dfrac{dT}{dt} + \alpha T = F(t)$,

where $f(t,T) = -\left(4\times10^{-12}\right)\times\left(T^4 - 250^4\right)$,

$f_0 = -\left(4\times10^{-12}\right)\times\left(2500^4 - 250^4\right) \;= -156.2343$,

$\left.\dfrac{\partial f}{\partial t}\right|_{t_0,T_0} \quad t_0 = 0$,

$\left.\dfrac{\partial f}{\partial T}\right|_{t_0,T_0} \quad T_0 = -\left(4\times10^{-12}\right)\times\left(4\times2500^3\right)\times2500 = -625.0000$,

$\left.\dfrac{\partial f}{\partial t}\right|_{t_0,T_0} \quad t = 0$

i.e. $F(t) = -156.2343 + 625.0000 = 468.7657$,

$\alpha = -\left.\dfrac{\partial f}{\partial T}\right|_{t_0,T_0} \; = -\left[-\left(4\times10^{-12}\right)\times\left(4\times2500^3\right)\right] = 0.25.$

$\therefore$ The linearised ODE will be $\dfrac{dT}{dt} + 0.25T = 468.7657$