CE 601: Numerical Methods

Lecture 25

Runge-Kutta Methods-I

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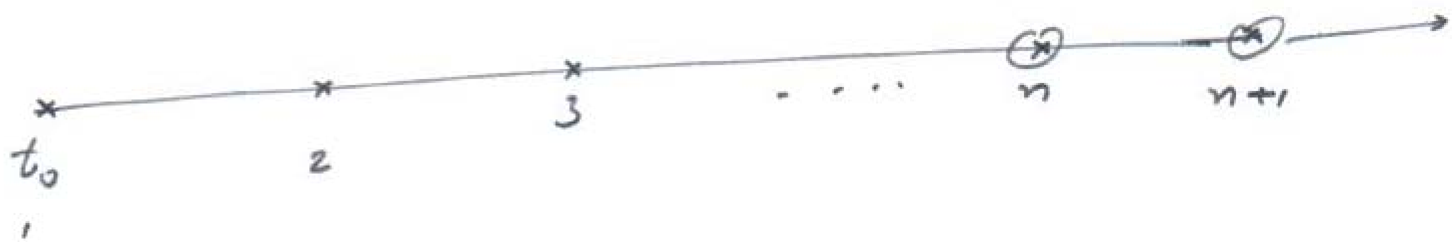
- In the last class we discussed on second-order Euler methods to solve initial-value non-linear ODE

$$\frac{dy}{dt} = f(t, y)$$

- In general the finite-difference method tries to solve or march forward to find the values in the next time steps.



$$y_{n+1} = y_n + \Delta t \times f$$

For explicit, $y_{n+1} = y_n + \Delta t \times f_n$

For implicit, $y_{n+1} = y_n + \Delta t \times f_{n+1}$

We can easily say now,

$$y_{n+1} - y_n = \Delta y$$

or, $y_{n+1} = y_n + \Delta y$

where $\Delta y \rightarrow$ change in $'y'$.

In Runge-Kutta methods, this change in Δy can be defined as sum of several weighted Δy's

i.e. $\Delta y = C_1 \times \Delta y_1 + C_2 \times \Delta y_2 + \cdots + C_m \times \Delta y_m$

where $C_i \rightarrow$ weighing factors,

$$\Delta y_i = \Delta t \times f(t, y)$$

[$f(t, y)$ is evaluted at some point in the range $t_n \leq t \leq t_{n+1}$]

$m \rightarrow$ suggestes the order of R-K method.

Second-order R-K Method is given as:

$$y_{(n+1)} = y_{(n)} + \left[C_1 \times \Delta y_1 + C_2 \times \Delta y_2\right] \qquad \rightarrow (1)$$

where $\Delta y_i = \Delta t \times f(t, y);\ t_n \le t \le t_{n+1}$

In second order R-K method it is assumed that

$$\Delta y_1 = \Delta t \times f_n$$

$$\therefore \Delta y_2 = \Delta t \times f(t, y);\ t_n \le t \le t_{n+1}$$

We need to suggest the suitable time between

t_n and t_{n+1} to evaluate derivative function $f(t, y)$

Let $\Delta y_2 = \Delta t \times f(t_n + a\Delta t, y_n + b\Delta y_1) \quad \rightarrow (2)$

We need to find 'a' and 'b'.

$$\therefore y_{(n+1)} = y_{(n)} + C_1 \times \Delta t \times f_n + C_2 \times \Delta t \times f(t_n + a\Delta t, y_n + b\Delta y_1)$$

Keeping the time t_n as the base time and f_n as base-value for function f, using Taylor's series

$$f(t,y) = f_n + \Delta t \times \left.\frac{\partial f}{\partial t}\right]_n + \Delta y \times \left.\frac{\partial f}{\partial y}\right]_n + \frac{(\Delta t)^2}{2!} \times \left.\frac{\partial^2 f}{\partial t^2}\right]_n + \frac{(\Delta y)^2}{2!} \times \left.\frac{\partial^2 f}{\partial y^2}\right]_n + \cdots$$

$$\therefore\ f(t_n + a\Delta t, y_n + b\Delta y_1) = f_n + (a\Delta t) \times \left.\frac{\partial f}{\partial t}\right]_n + (b\Delta y_1) \times \left.\frac{\partial f}{\partial y}\right]_n + \cdots; O(\Delta t^2) \quad \rightarrow (3)$$

Substituting (3) in (2), we get:

$$\Delta y_2 = \Delta t \times \left[f_n + (a\Delta t) \times \left.\frac{\partial f}{\partial t}\right]_n + (b\Delta y_1) \times \left.\frac{\partial f}{\partial y}\right]_n + \cdots \right]; \text{ Truncate } O(\Delta t^2).$$

$\therefore$ Eq. (1) becomes:

$$y_{n+1} = y_n + C_1 \times \Delta t \times f_n + C_2 \times \Delta t \times \left(f_n + (a\Delta t) \times \left.\frac{\partial f}{\partial t}\right]_n + (b\Delta y_1) \times \left.\frac{\partial f}{\partial y}\right]_n \right)$$

$$= y_n + \Delta t \times (C_1 + C_2) \times f_n + C_2 \Delta t^2 \times \left(a\Delta t \left.\frac{\partial f}{\partial t}\right]_n + b\Delta y_1 \left.\frac{\partial f}{\partial y}\right]_n \right)$$

Recall $\Delta y_1 = \Delta t \times f_n$

$$\therefore y_{n+1} = y_n + (C_1 + C_2)\Delta t \times f_n + C_2 \Delta t^2 \left[a \frac{\partial f}{\partial t}\bigg|_n + b f_n \frac{\partial f}{\partial y}\bigg|_n \right] \rightarrow (4)$$

We need to find suitable values of C_1, C_2, a and b for II$^{\text{nd}}$ order R-K.

Using Taylor's series to expand y_{n+1} with base point as y_n, we get:

$$y_{(n+1)} = y_{(n)} + \Delta t \times \frac{dy}{dt}\bigg]_n + \frac{(\Delta t)^2}{2!} \frac{d^2 y}{dt^2}\bigg]_n + \cdots$$

We know, $\frac{dy}{dt}\bigg]_n = f(t_n, y_n) = f_n$ and

$$\frac{d^2 y}{dt^2}\bigg]_n = \frac{d}{dt}\left(\frac{dy}{dt}\right)\bigg]_n = \frac{df}{dt}\bigg]_n = \frac{\partial f}{\partial t}\bigg]_n + \frac{\partial f}{\partial y}\bigg]_n \frac{dy}{dt}\bigg]_n = \frac{\partial f}{\partial t}\bigg]_n + f_n \frac{\partial f}{\partial y}\bigg]_n$$

$$\therefore y_{(n+1)} = y_{(n)} + \Delta t \times f_n + \frac{(\Delta t)^2}{2!}\left(\frac{\partial f}{\partial t}\bigg]_n + f_n \frac{\partial f}{\partial y}\bigg]_n \right) \rightarrow (5)$$

Comparing (4) and (5), we get,

$C_1 + C_2 = 1,$

$aC_2 = 1/2,$

$bC_2 = 1/2.$

There are only three equations and four unknowns.

There are many possibilities of C_1, C_2, a and b.

One such prominently used case is

put $C_1 = 1/2, \therefore C_2 = 1/2$ and $a = 1, b = 1.$

$$\therefore y_{n+1} = y_n + \frac{1}{2}\Delta y_1 + \frac{1}{2}\Delta y_2$$

i.e. $$\boxed{y_{n+1} = y_n + \frac{\Delta t}{2}\left[f_n + f_{n+1} \right].} \quad \rightarrow (6)$$

Eq. (6) is the Modified Euler's method.

Similarly you can develop higher order R-K formulas to solve IV-ODE.

The most famous is the 4^{th} Order R-K Method.

$$y_{n+1} = y_n + C_1\Delta y_1 + C_2\Delta y_2 + C_3\Delta y_3 + C_4\Delta y_4$$

where you can have: $\Delta y_1 = \Delta t \times f_n$

$$\Delta y_2 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_1}{2}\right)$$

$$\Delta y_3 = \Delta t \times f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_2}{2}\right)$$

$$\Delta y_4 = \Delta t \times f\left(t_n + \Delta t, y_n + \Delta y_3\right)$$

$$y_{n+1} = y_n + \frac{1}{6}\left(\Delta y_1 + 2\Delta y_2 + 2\Delta y_3 + \Delta y_4\right)$$

i.e. $$y_{n+1} = y_n + \frac{\Delta t}{6}\left[\begin{array}{l} f_n + 2f\left(t_{n+1/2}, y_n + \frac{\Delta y_1}{2}\right) + 2f\left(t_{n+1/2}, y_n + \frac{\Delta y_2}{2}\right) \\ + f\left(t_n + \Delta t, y_n + \Delta y_3\right) \end{array}\right]$$