CE 601: Numerical Methods

Lecture 24

IV-ODE:
Second Order Euler Methods

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- In last class we suggested that FDM, whichever you use, should be: Consistent, Stable and Convergent.
- We have already showed that first order explicit Euler scheme is consistent.
- What is meant by FDM being convergent?
- A FDM is said to be convergent if the solutions obtained from the finite difference algebraic equations approach the exact solution of the ODE as $\Delta t \to 0$.
- A FDM is stable if the finite difference algebraic equation gives bounded solution for a stable ODE.

- $\frac{dy}{dt} + \alpha y = 0$
- Let us check for the stability of the explicit Euler method solution of the above equation.
- As $f = -\alpha y$, hence

$$y_{n+1} = y_n + \Delta t.(-\alpha y_n)$$

$$\Rightarrow y_{n+1} = y_n (1 - \Delta t.\alpha)$$

- Hence, for stability, we require $-1 \leq (1 - \Delta t.\alpha) \leq 1$

In general: $y_{n+1} = Gy_n$

where $G \rightarrow$ amplification factor. In this case, $G = 1 - \alpha\Delta t$.

If y_0 is the initial value and if we have N cycles, the total time $T = N\Delta t$.

Then the solution at T will be

$$y_N = Gy_{N-1} = G^2 y_{N-2} = \cdots = G^N y_0$$

For y_N to remain bounded as $N \rightarrow \infty,$ we need to have $|G| \le 1.0.$ (Also $y_{n+1} = G^{n+1} y_0$)

If we utilise implicit Euler scheme for $\dfrac{dy}{dt} + \alpha y = 0,$

i.e. $y_{n+1} = \dfrac{y_n}{1 + \alpha\Delta t} = Gy_n;$ where $G = \dfrac{y_n}{1 + \alpha\Delta t} \le 1.0$

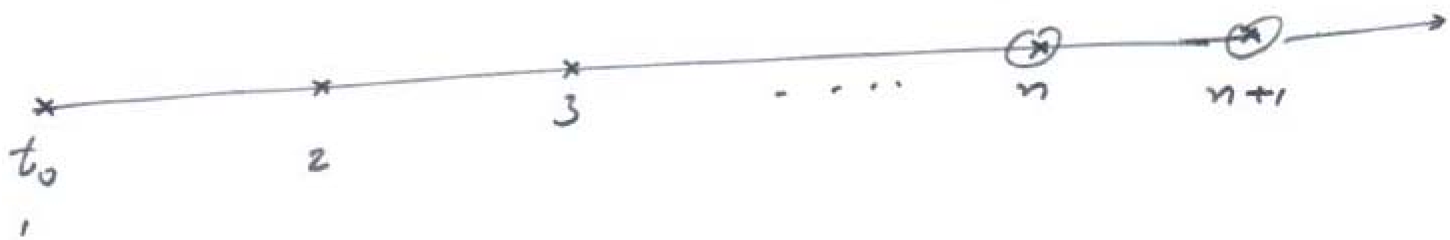
The Single Point Methods

- Both the first order Euler's explicit scheme and implicit schemes are single-point methods to arrive at the solution.
- In single point methods, we use data at a single point 'n' to advance to the solution at point $n+1$.
- The earlier mentioned methods were first-order schemes.
- We can also have second order single point methods.

- <u>Second order single point method</u>
- For the general non-linear IV-ODE (first order)

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

- The time scale can be discretized as:



Recall the centered difference formula

$$\left.\frac{dy}{dt}\right|_{t_{n+1/2}} \approx \frac{y_{n+1} - y_n}{\Delta t} \quad O(\Delta t^2)$$

$$\therefore \left.\frac{dy}{dt}\right|_{t_{n+1/2}} = f(t_{n+1/2}, y_{n+1/2}) = f_{n+1/2}$$

Using FDM, we get,

$$\boxed{y_{n+1} = y_n + \Delta t . f_{(n+\frac{1}{2})};} \qquad O(\Delta t^3)$$

$\rightarrow$ (1)

This is mid-point method.

This approach is difficult, as we need to determine the unknown $f_{n+1/2}$ as well.

A modification is suggested to mid-point method by a predictor-corrector strategy:

i.e. $$\boxed{\begin{aligned} y^P_{n+1/2} &= y_n + \frac{\Delta t}{2} \cdot f_n \\ y^C_{n+1} &= y_n + \Delta t . f^P_{n+1/2} \end{aligned}} \rightarrow (2)$$

This is a predictor-corrector method (also called Modified Mid-point method)

- Now recall the mid-point method

$$y_{n+1} = y_n + \Delta t . f_{n+\frac{1}{2}};$$

- The quantity $f_{n+\frac{1}{2}}$ can also be evaluated using the following strategy,
- Keeping $f_{n+\frac{1}{2}}$ as base point and form Taylor's series for f_n and f_{n+1}

$$f_n = f_{(n+\frac{1}{2})} - \left(\frac{\Delta t}{2}\right)\frac{df}{dt}\bigg|_{t_{(n+1/2)}} ; \qquad O(\Delta t^2)$$

$$f_{n+1} = f_{(n+\frac{1}{2})} + \left(\frac{\Delta t}{2}\right)\frac{df}{dt}\bigg|_{t_{(n+1/2)}} ; \qquad O(\Delta t^2)$$

- From the above two equations, we have

$$f_{(n+1/2)} = \frac{1}{2}\cdot\left(f_n + f_{n+1}\right); \qquad O(\Delta t^2)$$

- Hence,

$$\boxed{y^{(n+1)} = y^n + \frac{\Delta t}{2}\cdot\left(f_n + f_{n+1}\right); \qquad O(\Delta t^3)} \rightarrow (3)$$

- This is the implicit trapezoidal finite-difference equation.

- This can also be evaluated using the following predictor and corrector steps

$$
\boxed{\begin{aligned} y^{P}_{n+1} &= y_n + \Delta t . f_n \\ y^{C}_{n+1} &= y_n + \Delta t . \left(\frac{f_n + f^{P}_{n+1}}{2} \right) \end{aligned}} \rightarrow (4)
$$

- This equations are called modified trapezoidal method or Modified Euler's finite difference equation.
- Modified Euler's FDE is widely used to solve IV-ODE and the method is second order that suggests the truncation error reduces at faster rate.

- <u>Stability Criteria for Modified Euler Method</u>
- For the linear first order IV-ODE

$$\frac{dy}{dt} + \alpha\, y = 0; \qquad y(t_0) = y_0$$

- Modified Euler method

$$y_{n+1}^{P} = y_n + \Delta t . f_n$$

$$y_{n+1}^{C} = y^{n} + \left(\Delta t/2\right).\left(f_n + f^{P}{}_{n+1}\right)$$

- Here, we have

$$f = -\alpha\, y, \quad \therefore \; f_n = -\alpha\, y_n$$

- $$y_{n+1}^{P} = y_n + \Delta t.(-\alpha\, y_n) = \left(1-\alpha\Delta t\right) y_n$$

$$y_{n+1}^{C} = y_n - \alpha\left(\Delta t/2\right) y_n - \alpha\left(\Delta t/2\right) y_{n+1}^{P}$$

$$= y_n - \alpha\left(\Delta t/2\right) y_n - \alpha\left(\Delta t/2\right)\left\{ y_n.(1-\alpha\Delta t)\right\}$$

$$= y_n - \alpha\left(\Delta t/2\right) y_n - \alpha\left(\Delta t/2\right) y_n + \alpha^2.\left(\Delta t^2/2\right) y_n$$

$$= y_n\left\{1-\alpha\Delta t + \left(\alpha\Delta t\right)^2/2\right\}$$

- Hence, the amplification factor:

$$G = \left\{1-\alpha\,\Delta\,t + \left(\alpha\,\Delta\,t\right)^2/2\right\}$$

- For stable results, the following condition has to be satisfied

$$|G| \leq 1.0$$

- So, $|G| = \left|1 - \alpha\Delta t + (\alpha\Delta t)^2 / 2\right| \leq 1.0$
- For the above condition to be true, we require

$$\alpha\Delta t \leq 2.0$$

Runge-Kutta Methods

- Till now we discussed about the various second order single point Euler methods to solve the general first order non-linear IV-ODE

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

- Runge-Kutta methods are single point methods that evaluate change in *y* i.e. $\Delta y = y_{n+1} - y_n$ using several weighted combinations of Δy_i

$$\text{i.e. } y_{n+1} = y_n + \Delta t f_n$$

$$\text{or}$$

$$y_{n+1} = y_n + \Delta t f_{n+1}$$

We can define $\Delta y = y_{n+1} - y_n$

or, $\Delta y = \Delta t f_n$

or

$\Delta y = \Delta t f_{n+1}$

or

etc.

This change in y,

$$\Delta y = C_1 \times \Delta y_1 + C_2 \times \Delta y_2 + C_3 \times \Delta y_3 + \cdots + C_n \times \Delta y_n$$

$C_i \rightarrow$ weighting factors

$\Delta y_i = \Delta t \times f(t, y);$ where $f(t, y)$ is evaluated at some point in the range $t_n \leq t \leq t_{n+1}$.

$\Rightarrow$ The number of weighing factors (or no. of Δy_i) selected decides the order of the R-K method.

$\Rightarrow$ e.g. The first order R-K method will be same as explicit or implicit Euler method

$$y_{n+1} = y_n + \Delta t \times f_n$$

or

$$y_{n+1} = y_n + \Delta t \times f_{n+1}$$

$\Rightarrow$ The second order R-K method is:

$$y_{n+1} = y_n + \left[C_1 \Delta y_1 + C_2 \Delta y_2\right]$$

where $\Delta y_1 = \Delta t \times f_n$

$$\Delta y_2 = \Delta t \times f(t, y); \quad t_n \le t \le t_{n+1}$$

Let's write: $\Delta y_2 = \Delta t \times f(t_n + a\Delta t, y_n + b\Delta y_1)$

We have to find the appropriate values of 'a' and 'b' for this case.

$\Rightarrow 2^{nd}$ order R-K method gives:

$$y_{n+1} = y_n + C_1 \times \Delta t \times f_n + C_2 \times \Delta t \times f(t_n + a\Delta t, y_n + b\Delta y_1)$$

Keeping the time grid point 'n' as the base point and

f_n as base value,

$$f(t, y) = f_n + (t - t_n)\left.\frac{\partial f}{\partial t}\right]_{t_n} + (y - y_n)\left.\frac{\partial f}{\partial y}\right]_{t_n} + \cdots$$

Here $t = t_n + a\Delta t$ and $y_n + b\Delta y_1$,

$$\therefore\ f(t_n + a\Delta t, y_n + b\Delta y_1) = f_n + \left(a\Delta t\right)\left.\frac{\partial f}{\partial t}\right]_{t_n} + \left(b\Delta y_1\right)\left.\frac{\partial f}{\partial y}\right]_{t_n} + \cdots\ O\left(\Delta t^2\right)$$