CE 601: Numerical Methods

Lecture 23

IV-ODE:
Finite Difference Method

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Initial Value ODE’s

- In the last class, we have introduced about Ordinary Differential Equations
- Classification of ODEs:
- Based on the conditions given to the application of an ODE, they can be classified as
  - Initial value ODE
  - Boundary value ODE
- The IV-ODE’s mostly describe propagation problems.
- The BV-ODE’s mostly describe equilibrium problems

- In Initial value ODE's

❖ A general linear first-order ODE is

$$\frac{dy}{dt} + \alpha y = F(t); \quad y(t_0) = y_0$$

where $\alpha \to \alpha(t)$ or constant.

❖ A general non-linear first-order ODE is

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

- <u>To solve IV-ODE's using Finite difference method:</u>
- Objective of the finite difference method (FDM) is to convert the ODE into algebraic form.
- The following steps are followed in FDM:
  - Discretize the continuous domain (spatial or temporal) to discrete finite-difference grid.
  - Approximate the derivatives in ODE by finite difference approximations.
  - Substitute these approximations in ODEs at any instant or location.
  - Obtain algebraic equations.
  - Solve the resulting algebraic equations or Finite Difference Equations (FDE).

- We have seen on the last class, how the forward, backward and centered finite difference formulas can derive different finite-difference equations.
- i.e. for the general non-linear first order IV-ODE:

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

$$\text{Then } \left.\frac{dy}{dt}\right|_{t_n} \approx \frac{y_{n+1} - y_n}{\Delta t} \quad \text{(Forward difference)}$$

$$\left.\frac{dy}{dt}\right|_{t_{n+1}} \approx \frac{y_{n+1} - y_n}{\Delta t} \quad \text{(Backward difference)}$$

$$\left.\frac{dy}{dt}\right|_{t_{n+1/2}} \approx \frac{y_{n+1} - y_n}{\Delta t} \quad \text{(Centered difference)}$$

As $\dfrac{dy}{dt} = f(t, y)$

$$\left.\frac{dy}{dt}\right|_{t_n} = f(t_n, y_n) = f_n$$

$$\left.\frac{dy}{dt}\right|_{t_{n+1}} = f(t_{n+1}, y_{n+1}) = f_{n+1}$$

and $\left.\dfrac{dy}{dt}\right|_{t_{n+1/2}} \approx \dfrac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1/2}, y_{n+1/2}) = f_{n+1/2}$

$\therefore y_{n+1} = y_n + \Delta t f_n$ (Explicit finite-difference eqn.)

$y_{n+1} = y_n + \Delta t f_{n+1}$ (Implicit finite-difference eqn.)

- Care should be taken that the functions involved in FDM solutions are continuous and smooth.
- Else, it can give error or fluctuation.
- While using FDM, following errors can creep:
  - Error in initial data
  - Algebraic errors
  - Truncation errors
  - Round off errors
  - Inherited errors
  - Errors due to faulty formulations

First Order Euler Methods $\left(\text{For } \frac{dy}{dt} = f(t,y);\ y(t_0) = y_0 \right)$

1) The Explicit Euler Method

The explicit FDE is

$$y_{n+1} = y_n + \Delta t f_n;\ O(\Delta t^2)$$

Here the truncation error reduces at a speed $O(\Delta t^2)$.

2) The Implicit Euler Method

$$\_\_\_\_\_\_ \underset{\otimes}{n} \_\_\_\_\_\_ \underset{\otimes}{n+1} \_\_\_\_\_\_$$

$$t \rightarrow$$

$$\left.\frac{dy}{dt}\right|_{t_{n+1}} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

The FDE is:

$$y_{n+1} = y_n + \Delta t f_{n+1};\ O(\Delta t^2)$$

- <u>Comparison of explicit and implicit methods:</u>
- It can be seen that the explicit method gives the solution directly.
- So, what is the need to go for implicit methods?
- Consider a homogeneous initial value ODE:

$$\frac{dy}{dt} + y = 0$$

- We already know that the solution of this equation is $y = e^{-t}$
- Using FDM, we will see how the solution appears.

- Using explicit Euler method

$$y_{n+1} = y_n + f_n \Delta t$$

$$f_n = -y_n$$

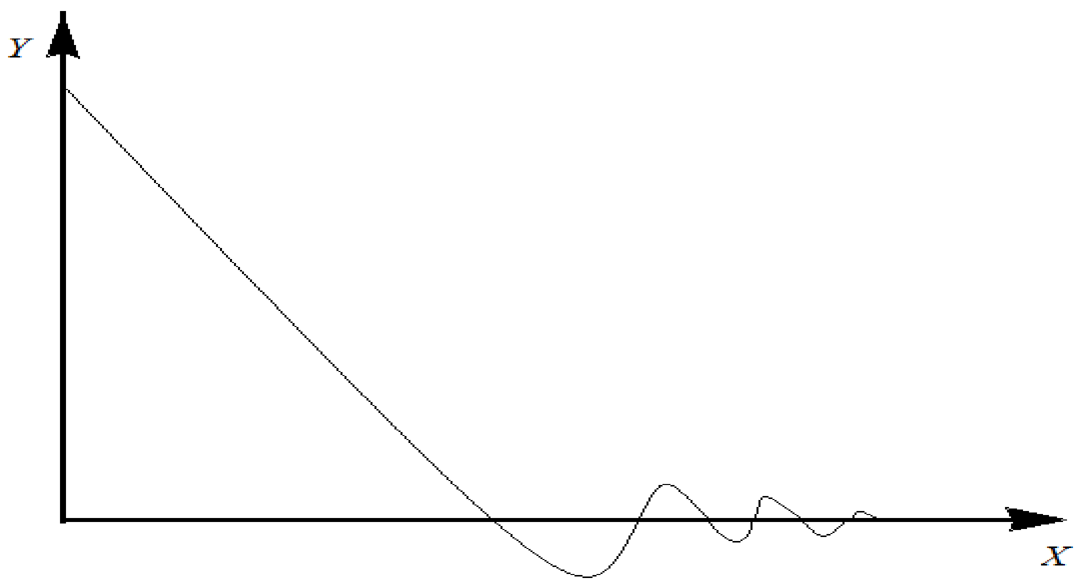
- Hence

$$y_{n+1} = y_n - y_n \Delta t$$

$$\Rightarrow y_{n+1} = y_n (1 - \Delta t)$$

- We have seen that the initial value of the function is $y = 1.000$ at $t = 0.000$
- It can be seen that for $\Delta t \leq 1.0$, the aforementioned expression for y_{n+1} has realistic values.
- For $2.0 \geq \Delta t \geq 1.0$, it can be seen that the sign of the solution changes after each time step.

- The solution would fluctuate around zero, but converges to zero at large values of t.



- For $\Delta t > 2.0$, the solution diverges and hence, we will not get stable solutions.

- Using Implicit Euler method

$$y_{n+1} = y_n + f_{n+1}\Delta t$$

$$y_{n+1} = y_n - y_{n+1}\Delta t$$

- or, $$y_{n+1} = \frac{y_n}{(1+\Delta t)}$$

- It can be seen that even if $\Delta t > 1.0$, the solution is available for the ODE.
- The implicit method happens to be unconditionally stable.

- Requirements for Finite Difference Methods:
- To successfully solve given ODEs, the FDMs should be:
  - Consistent
  - Stable
  - Convergent

- What is meant by a FDM being consistent?
- It means that the difference between the finite difference algebraic equation and the original ODE vanishes when $\Delta t \to 0$.
- Consider the following Initial value ODE:

$$\frac{dy}{dt} + \alpha y = F(t); \qquad y(t_0) = y_0$$

- Using explicit Euler method

$$y_{n+1} = y_n + f_n \Delta t$$

- Here, $f_n = F(t_n) - \alpha y_n$

$$\left.\frac{dy}{dt}\right|_{t_n} + \alpha y^n = F(t_n)$$

- Since in this case $f_n = F_n - \alpha y_n$
- So,

$$y^{(n+1)} = y^n + F_n \Delta t - \alpha \Delta t . y^n \quad (1)$$

- Considering y_n as base point, Taylor's series is utilized to evaluate y_{n+1}

$$y_{n+1} = y_n + \Delta t.\left.\frac{dy}{dt}\right|_{t_n} + \frac{(\Delta t)^2}{2!}.\left.\frac{d^2y}{dt^2}\right|_{t_n} + \cdots\cdots\cdots\cdots\cdots \quad (2)$$

From the equations (1) and (2), we have

$$\Delta t.\left.\frac{dy}{dt}\right|_{t_n} + \frac{(\Delta t)^2}{2!}.\left.\frac{d^2y}{dt^2}\right|_{t_n} + \cdots\cdots = F_n \Delta t - \alpha \Delta t.y^n$$

- Dividing the previous equation by Δt , we have
- $$\left.\frac{dy}{dt}\right|_{t_n} + \frac{\Delta t}{2!} \cdot \left.\frac{d^2 y}{dt^2}\right|_{t_n} + \cdots\cdots = F_n - \alpha . y^n$$
- As $\Delta t \to 0$, we get
  $$\left.\frac{dy}{dt}\right|_{t_n} + \alpha y^n = F_n$$
- So, the differential equation and FDE at $\Delta t \to 0$ are same. Hence, explicit Euler method is consistent.