Lecture 22

Ordinary Differential Equations

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

- In nature, most of the phenomena that can be mathematically described are in the form of differential equations.
- For example, in hydrology, you may have studied the classic conservation equation

$$\frac{dS}{dt} = I(t) - Q(t)$$

where S is the water in the reservoir or water resource system

$I(t)$ is the inflow to the system

$Q(t)$ is the outflow from the system

- Similarly, there are many phenomena that can be represented in differential equation form.
- Definition of differential equation:
- It is an equation consisting of derivatives of a dependent variable with respect to independent variable(s).

- If a dependent variable is varying with respect to more than one independent variable, then the governing equations formed are usually partial differential equations.
- If in certain situations, we can approximate the dependent variable to be varying with respect to only one independent variable, then, the equation is an ordinary differential equation.

- $\frac{dy}{dt} = f(t, y)$

- What is the order of the above ODE?
- Order of an ODE is the highest order derivative present in that ODE.
- A general nth order ODE can be represented as

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dt^{n-2}} + \cdots\cdots\cdots\cdots\cdots\cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = F(t)$$

- What is a linear ODE?
- An ODE in which all the derivatives appear in linear form and the coefficients do not depend on the dependent variable is known as a linear ODE.
- $\frac{dy}{dt} + \alpha y = f(y)$
- where α is either a constant coefficient or a function of t.
- However, if $\alpha = \alpha(t, y)$, then the above equation is a non linear ODE.

- Homogeneous ODE

$$\frac{dy}{dt} + \alpha y = 0$$

- Non-Homogeneous ODE

$$\frac{dy}{dt} + \alpha y = f(t)$$

- System of ODE

$$\frac{dx}{dt} = f(x, y, z, t)$$

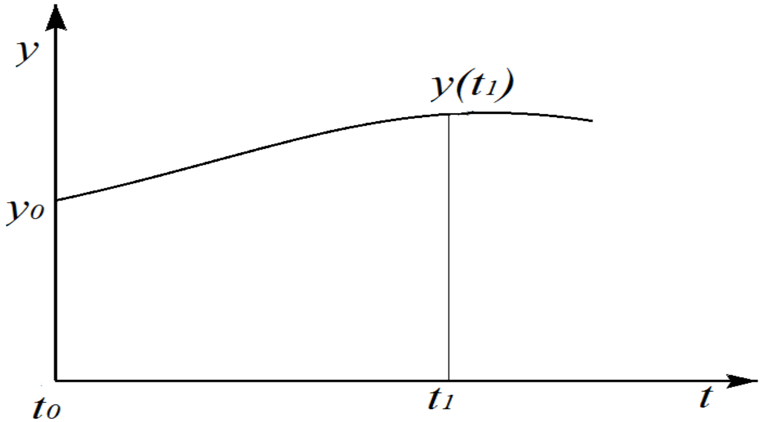
$$\frac{dy}{dt} = g(x, y, z, t)$$

$$\frac{dz}{dt} = h(x, y, z, t)$$

- Discussion in this lecture shall be pertaining to
- General non-linear first order ODE
- $\frac{dy}{dt} = f(t, y)$
- General non-linear second order ODE
- $\frac{d^2 y}{dt^2} + P(y,t)\frac{dy}{dt} + Q(y,t)y = f(t)$

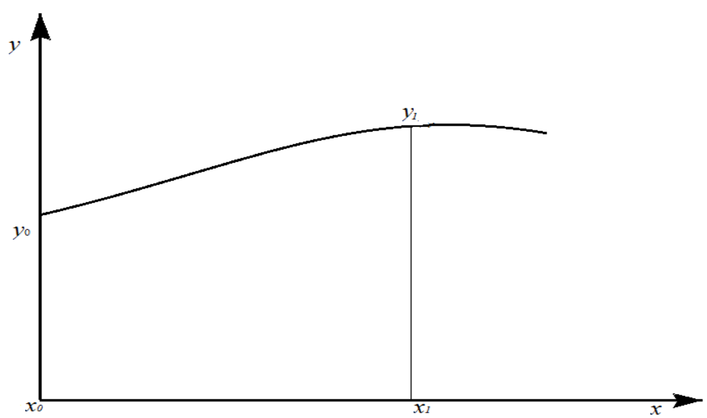
- Classification of ODEs:
- Based on the conditions given to the application of an ODE, they can be classified as
  - Initial value ODE
  - Boundary value ODE

- Initial value ODE



$$\frac{dy}{dt} = f(t, y); \qquad y(t_0) = y_0$$

- Boundary Value ODE



- Actual classification of ODE problems:
- Propagation problems
- Equilibrium problems
- Eigen problems

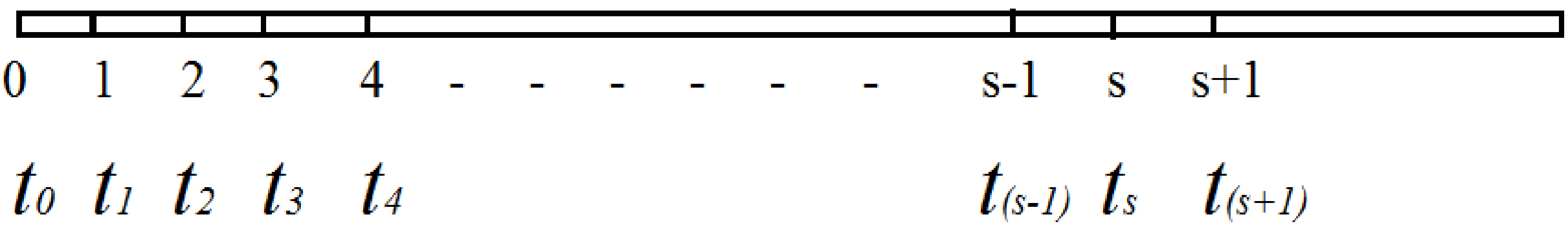
- One Dimensional Initial value ODE:
- As has been described earlier, most of the governing equations are differential equations (ODE and PDE).
- Some of the simplified cases can be described using ODEs.
- For example, the heat transfer to the surrounding is described using the first order ODE:

$$\frac{dT}{dt} = \alpha \;\; T^4 - T_0^4 \;\;, \qquad\qquad T(0) = T_0$$

- This is an initial value ODE for which initial condition has been mentioned.

- Finite difference method:
- Objective of the finite difference method (FDM) is to convert the ODE into algebraic form.
- The following steps are followed in FDM:
  - Discretize the continuous domain (spatial or temporal) to discrete finite-difference grid.
  - Approximate the derivatives in ODE by finite difference approximations.
  - Substitute these approximations in ODEs at any instant or location.
  - Obtain algebraic equations.
  - Solve the resulting algebraic equations.

- Discretization of temporal domain:



$$\left.\frac{dy}{dt}\right|_{t_s} = f(y_s, t_s)$$

- $$\left.\frac{dy}{dt}\right|_{t_s} \approx \frac{y_{(s+1)} - y_s}{\Delta t} \qquad \mathrm{O}(\Delta t)$$ (Forward difference)

- $$\left.\frac{dy}{dt}\right|_{t_s} \approx \frac{y_s - y_{(s-1)}}{\Delta t} \qquad \mathrm{O}(\Delta t)$$ (Backward difference)

- $$\left.\frac{dy}{dt}\right|_{t_s} \approx \frac{y_{(s+1)} - y_{(s-1)}}{2\Delta t} \qquad O(\Delta t^2) \qquad \text{(Central difference)}$$

- Substituting the value of the derivative according to the forward difference scheme in the differential equation, we have

$$\frac{y_{(s+1)} - y_s}{\Delta t} = f(y_s, t_s)$$

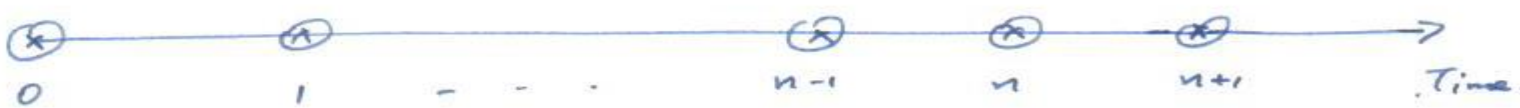
or

$$y_{(s+1)} = y_s + \Delta t \, f(y_s, t_s)$$

- This is a finite difference algebraic equation.

- Care should be taken that the functions involved in FDM solutions are continuous and smooth.
- Else, it can give error or fluctuation.
- While using FDM, following errors can creep:
  - Error in initial data
  - Algebraic errors
  - Truncation errors
  - Round off errors
  - Inherited errors

- First order approximations for $\dfrac{dy}{dt} = f(t, y)$
- For $\dfrac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$



$$\left.\frac{dy}{dt}\right|_{t^n} = f(t^n, y^n)$$

- If we use forward difference formula, we will have $\left.\dfrac{dy}{dt}\right|_{t^n} = \dfrac{y^{(n+1)} - y^n}{\Delta t}$

- So, from the equation, $f_n = \frac{y^{(n+1)} - y^n}{\Delta t}$ where $f_n = f(t^n, y^n)$
- So, $y^{(n+1)} = y^n + f_n \Delta t;$ $O(\Delta t^2)$
- This is known as first order explicit Euler method.
- On repetitive application of Euler explicit method, the order of approximation reduces to $O(\Delta t)$

- Implicit Euler Method:

$$\left.\frac{dy}{dt}\right|_{n+1} = \frac{y^{(n+1)} - y^{n}}{\Delta t}$$

- Since $\left.\frac{dy}{dt}\right|_{n+1} = f\,(t^{n+1}, y^{n+1})$

$$y^{(n+1)} = y^{n} + f\,(t^{n+1}, y^{n+1})\Delta t; \qquad O(\Delta t^{2})$$

- This is the first order implicit Euler method.