CE 601: Numerical Methods

Lecture 21

Numerical Integration

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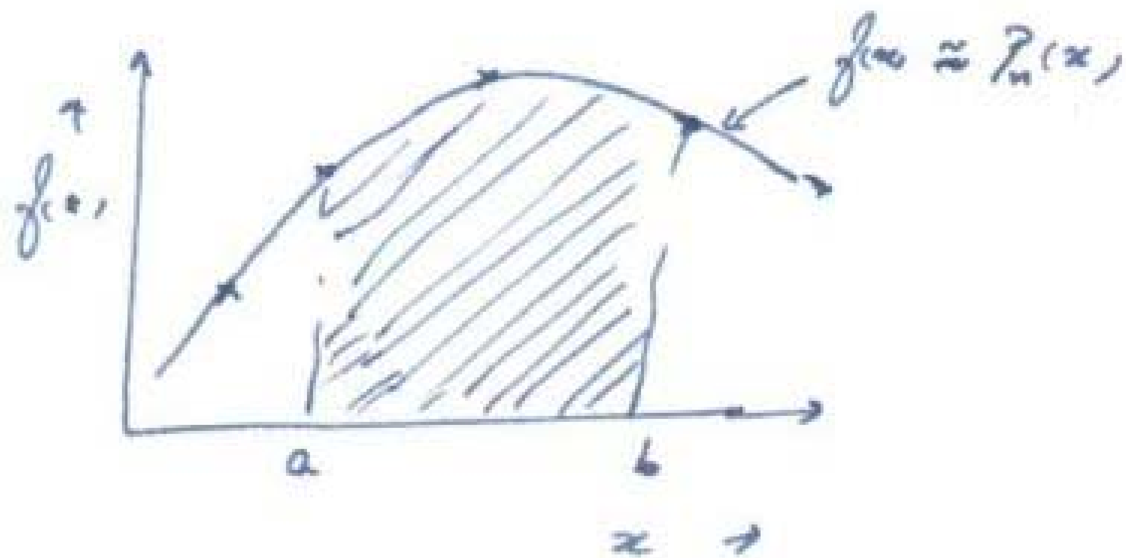
Numerical Integration

- For a given set of discrete data *(x_i, f_i),* we have seen that we can develop a function relation of '*f*' w.r.t. '*x*' using polynomial approximations.
  - → We also said that such polynomials should be able to do
    - ❖ Interpolation (already seen)
    - ❖ Differentiation (numerical differentiation)
    - ❖ Integration
  - → The numerical integration schemes will be discussed here.
- If we want to do, $I = \int_a^b f(x)dx$

Then bound on the data set, we had formed

$$f(x) \approx P_n(x), \therefore I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx.$$



• How the polynomials are constructed, you have already seen.

o Using direct-fit polynomials

$$I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \int_a^b \left(a_0 + a_1 x + \cdots + a_n x^n\right)dx$$

o Similarly Lagrange polynomials, Divided difference, Newton’s polynomials, etc. can be integrated.

o The evaluation of integrals of functions using such polynomials are called <u>quadratures</u>.

• Newton – Cotes Quadratures

Recall the Newton's forward difference polynomial for uniformly spaced data

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \cdots + \frac{s(s-1)(s-2)\cdots(s-(n-1))}{n!}\Delta^n f_0$$

$$\left[\text{Error} = \frac{s(s-1)(s-2)\cdots(s-n)}{(n+1)!}(\Delta x)^{n+1} f^{n+1}(\xi); x_0 \le x \le x_n \right]$$

$$\therefore I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

$$s = \frac{x - x_0}{\Delta x} \quad \text{or, } x = x_0 + s\Delta x$$

For $x = a,\ a = x_0 + s(a)\Delta x$

$x = b,\ b = x_0 + s(b)\Delta x.$

$$I \approx \Delta x \int_{s(a)}^{s(b)} P_n(s)ds.$$

If we choose the base point of the polynomial at $x = a$,

then $s = 0$ at $x = a$, and $s = \dfrac{x-a}{\Delta x}$

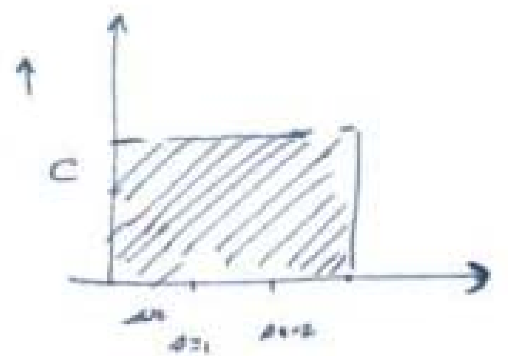
$$\text{at } x = b, s = \frac{b-a}{\Delta x}$$

$$\therefore I \approx \Delta x \int_0^s P_n(s) ds.$$

We can adopt various degrees of polynomials for our convenience.

1) If $n = 0$,

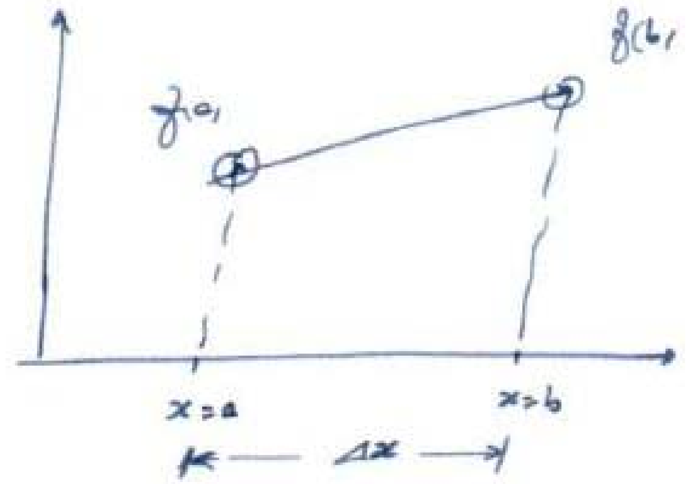
then $I = \Delta x \int_0^s P_0(s) ds$



$P_0 \rightarrow$ Some constant C,

$\boxed{I = \Delta x C s}$ This is rectangular formula for numerical integartion.

2) <u>If $n=1$</u>



$$I = \Delta x \int_0^s P_1(s)\,ds$$

$$= \Delta x \int_0^s \left(f_0 + s\Delta f_0 \right) ds$$

$$= \Delta x \left[sf_0 + \frac{s^2}{2}\Delta f_0 \right]$$

So if your data set is such a way that you used a first degree polynomial between $x=a$ and $x=b$ and if they are two consecutive points,

Then at $x=a,\ s=0$

$x=b,\ s=1,$ then: $\quad I = \Delta x \int_0^1 P_1(s)\,ds$

$$\therefore I = \Delta x \left[f_0 + \frac{1}{2}\Delta f_0 \right]$$

$$= \Delta x \left[f_0 + \frac{1}{2}\left(f_1 - f_0 \right) \right]$$

$I = \dfrac{\Delta x}{2}\left[f_0 + f_1 \right] \rightarrow$ This is trapezoidal rule to find area or integration, If linear splines are used to connect data from $x = x_0$ to $x = x_n$

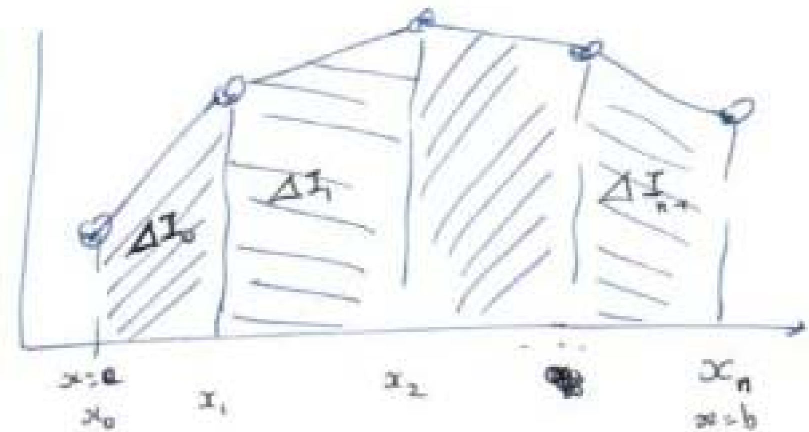
In such situations, the integration $I = \int_{x_0}^{x_n} f(x)dx$ will

be summations $I \approx \Delta I_0 + \Delta I_1 + \cdots + \Delta I_{n-1}$

Now $\Delta I_0 = \dfrac{\Delta x}{2}\left(f_0 + f_1 \right)$

Similarly, $\Delta I_1 = \dfrac{\Delta x}{2}\left(f_1 + f_2 \right)$

$$\therefore I = \sum_{i=0}^{n-1} \Delta I_i = \frac{1}{2}\sum_{i=0}^{n-1} \Delta x\left[f_i + f_{i+1} \right]$$



i.e. $\boxed{I = \frac{1}{2}\Delta x\left[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n \right]} \rightarrow$ Trapezoidal Rule

Q. What will be the order of approximation error for this trapezoidal rule?

Error for a single interval say $\Delta I_i, E_i = \Delta x \int_0^1 \left[\frac{s(s-1)}{2} \Delta^2 f_0 + \cdots \right] ds$

i.e. $E_i = \Delta x \int_0^1 \frac{s(s-1)}{2} \Delta x^2 f''(\zeta) ds$

$$= -\frac{1}{12} \Delta x^3 f''(\zeta) \rightarrow O(\Delta x^3)$$

For $I = \sum \Delta I_i$

Total error $= \sum_{i=0}^{n-1} E_i = \sum_{i=0}^{n-1} \left[-\frac{1}{12} \Delta x^3 f''(\zeta) \right]$

$$= -\frac{n}{12} \Delta x^3 f''(\zeta)$$

$$= -\left(\frac{x_n - x_0}{\Delta x} \right) \frac{1}{12} \Delta x^3 f''(\zeta) \rightarrow O(\Delta x^2)$$

That is, total integration is of $O(\Delta x^2)$

3) When $n = 2$ for Newton-Cotes integration

Recall $I = \Delta x \int_0^s P_n(s)ds$

If $n = 2,$ then second degree polynomial.

Minimize three points required.

If we choose three consecutive points $x = a = x_0, x = x_1, x = b = x_2,$

for $P_2(s)$ development, then at $x = a = x_0, s = 0$

at $x = x_1, s = 1$

at $x = b = x_2, s = 2.$

$$\therefore I = \Delta x \int_0^2 \left[f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 \right] ds$$

$$I = \Delta x\left[sf_0 + \frac{s^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)\Delta^2 f_0 \right]_0^2$$

$$= \Delta x\left[2f_0 + 2(f_1 - f_0) + \frac{4}{2}\left(\frac{2}{3} - \frac{1}{2}\right)(f_2 - 2f_1 + f_0) \right]$$

$$= \Delta x\left[\frac{1}{3}f_2 + \frac{4}{3}f_1 + \frac{1}{3}f_0 \right]$$

$$\boxed{I = \frac{\Delta x}{3}\left[f_2 + 4f_1 + f_0 \right]} \rightarrow \text{Simpson's } \frac{1}{3}^{\text{rd}} \text{ rule.}$$

→ This is only applied only for 1 interval between x_0 and x_2.

$\rightarrow$ If there are many intervals (especially while using splines),

$$I = \Delta I_0 + \Delta I_1 + \cdots + \Delta I_n$$

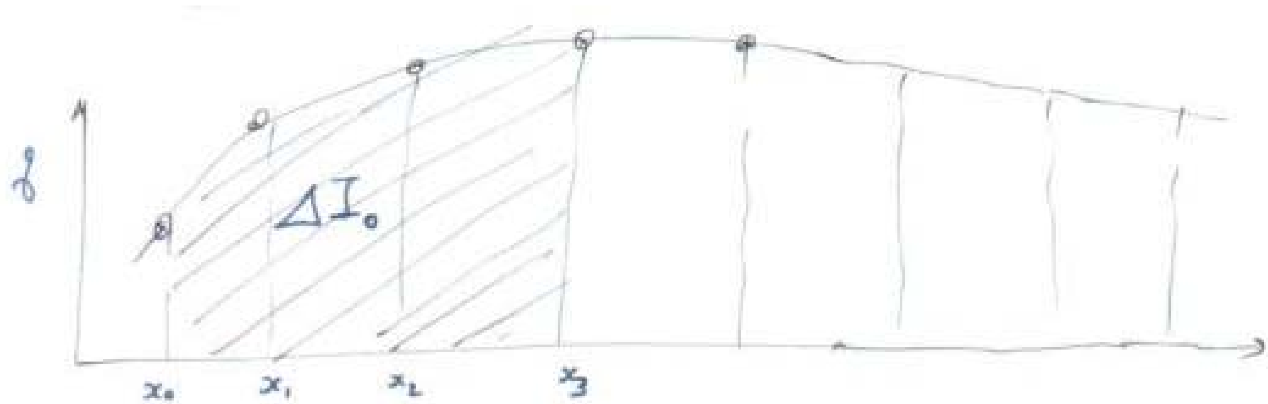
$$= \frac{\Delta x}{3}\left(f_0 + 4 f_1 + f_2 \right) + \frac{\Delta x}{3}\left(f_2 + 4 f_3 + f_4 \right) + \cdots + \frac{\Delta x}{3}\left(f_{n-2} + 4 f_{n-1} + f_n \right)$$

$$I = \frac{\Delta x}{3}\left[f_0 + 4 f_1 + 2 f_2 + 4 f_3 + 2 f_4 + \cdots \right]$$

$\rightarrow$ The number of data points should be odd and minimum 3.

Simpson's $\frac{1}{3}^{\text{rd}}$ rule for evaluating integration of data.

4) <u>When $n = 3$</u>



$$\Delta I_0 = \Delta x \int_0^s P_3(s)\,ds$$

$$= \Delta x \int_0^3 \left(f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 \right) ds$$

$$= \Delta x \left[s f_0 + \frac{s^2}{2}\Delta f_0 + \left(\frac{s^3}{6} - \frac{s^2}{4}\right)\Delta^2 f_0 + \left(\frac{s^4}{24} - \frac{s^3}{6} + \frac{s^2}{6}\right)\Delta^3 f_0 \right]_0^3$$

$$= \Delta x \left[\frac{3}{8} f_3 + \frac{9}{8} f_2 + \frac{9}{8} f_1 + \frac{3}{8} f_0 \right]$$

$$\boxed{\Delta I_0 = \frac{3}{8}\Delta x \left[f_0 + 3 f_1 + 3 f_2 + f_3 \right]} \rightarrow \text{Simpson's } \frac{3}{8}^{\text{th}} \text{ rule for numerical integration.}$$

So Total integration for $(n+1)$ data points using

Simpson's $\frac{3}{8}^{\text{th}}$ rule will be,

$$I = \Delta I_0 + \Delta I_1 + \cdots + \Delta I_n$$

$$= \frac{3}{8}\Delta x\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \cdots + 3f_{n-1} + f_n \right]$$

$\rightarrow$ The number of increments here should be multiples of three.

$\rightarrow$ Total data points $= 4 + 3k;$ where $k \rightarrow$ No. of intervals.

$\rightarrow$ The local and global error order can be inferred appropriately.