CE 601: Numerical Methods

Lecture 16

Cubic Splines

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

<u>Cubic Splines</u>

- Till now, we wee dealing with polynomials that approximate the actual function for a given data set.
- For a given (n+1) dat set, we were able to develop <u>exact-fit</u> polynomials.

$P_1(x)$ → if only two of the (n+1) data used

$P_2(x)$ → if only three of the (n+1) data used, etc.

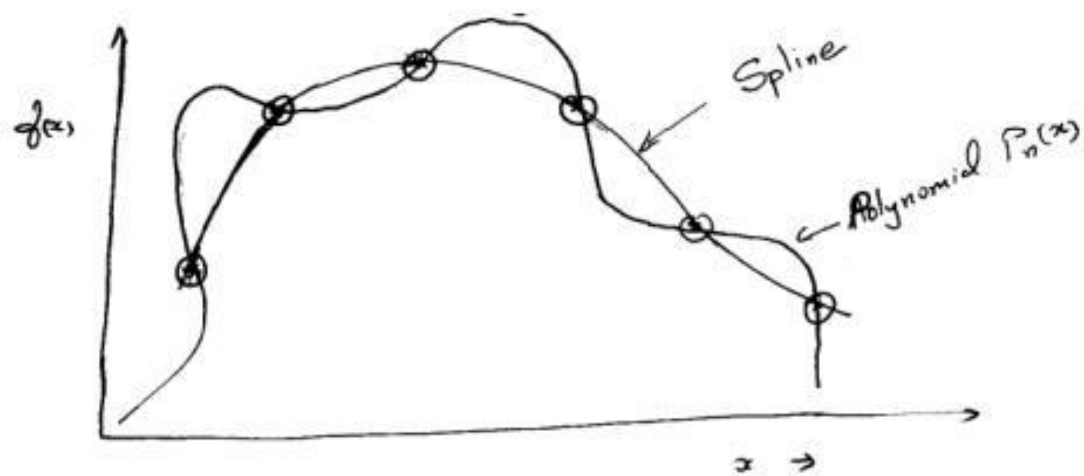
:

$P_n(x)$ → the unique exactly-fit polynomial that passes through all the given (n+1) data points.

- So we can see that for given data set, higher the polynomial order – greater the chance the polynomial passes through all the data points.
- Higher degree polynomials, therefore have few errors at the data points.

- What happens if we increase the degree of the polynomial.

x_i	f_i
x_0	f_0
x_1	f_1
x_2	f_2
$\vdots$	$\vdots$
x_n	f_n



- If we use higher degree polynomials it is passing through all the data points (as shown).
- However by looking into the shape, you may be having a feeling that in between data points, the polynomial may not be predicting the value accurately.

- Most of the times for higher degree polynomials this happens.
- This is true for most of the higher order polynomials.
- Therefore two questions remain in front of you:
- What is the point in going for higher degree polynomials?
- The lower degree polynomials will not pass through all the points.
- To overcome such issues, we can use splines.
- What is spline?
- Recall in your engineering graphics days, you used to take splines (or flexible rods) and use them to draw curves between multiple points.

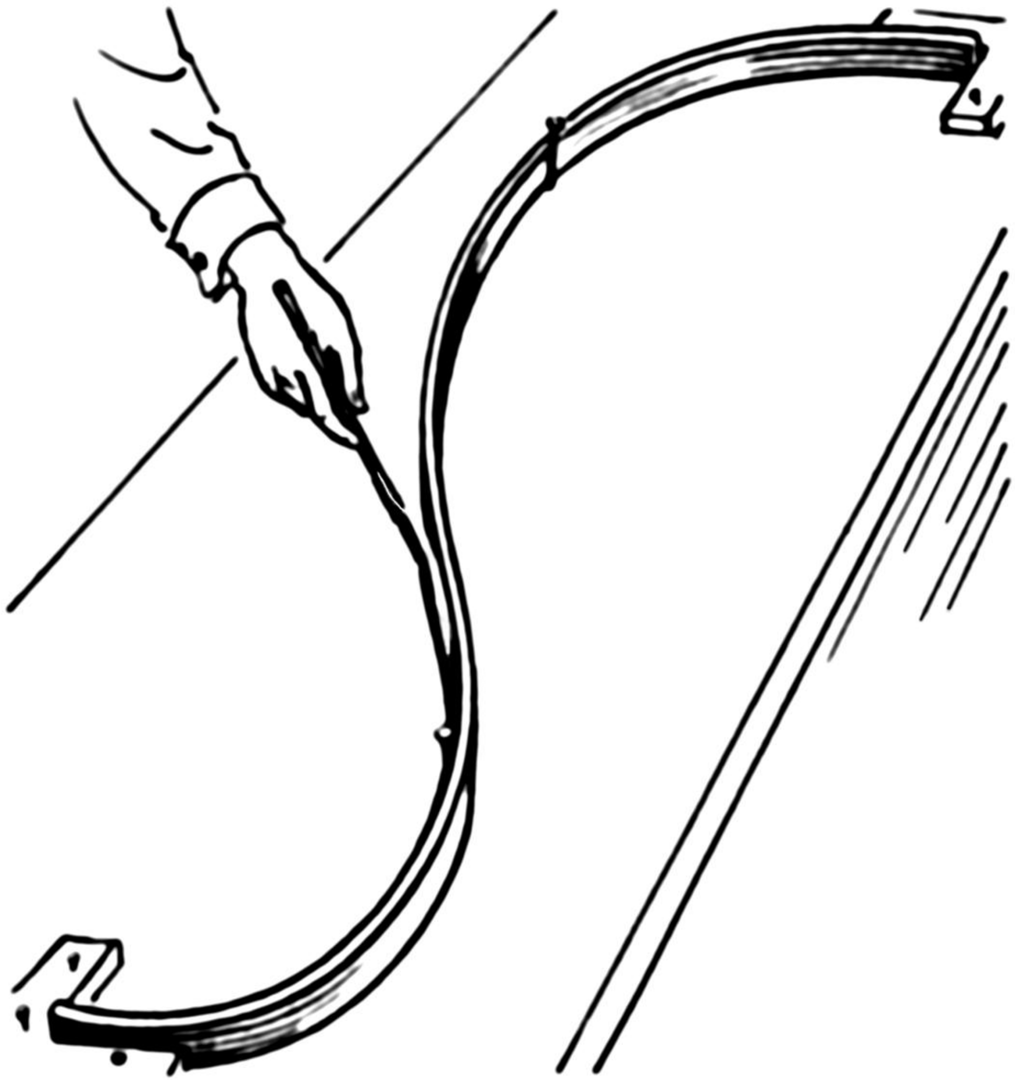
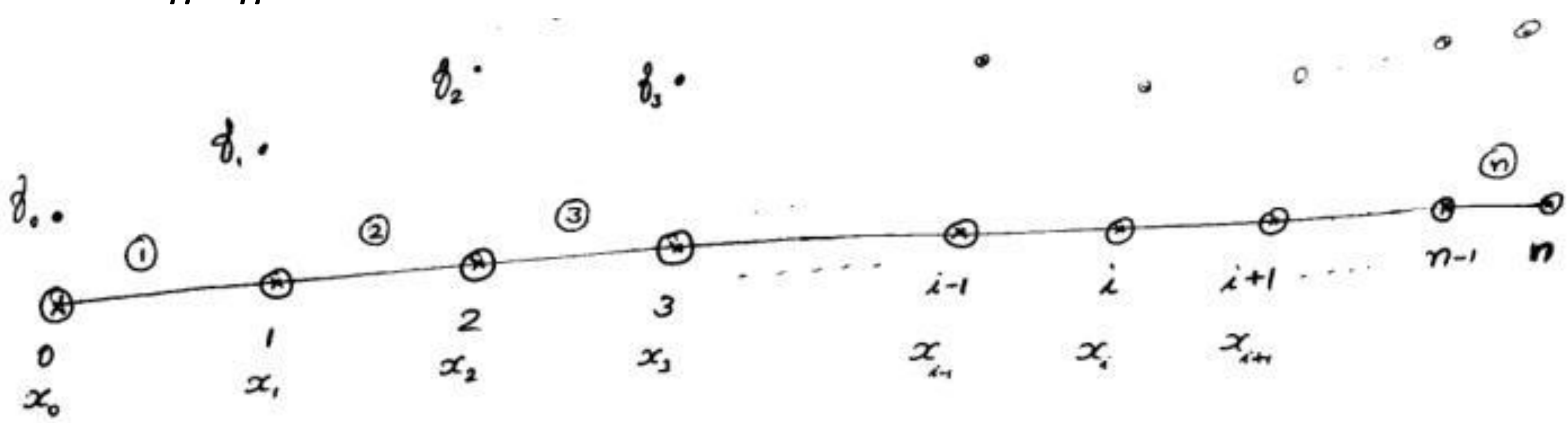


Fig. Spline

(ref: http://en.wikipedia.org/wiki/Flat_spline#mediaviewer/File:Spline_%28PSF%29.png

- The same philosophy is used here.
- You will connect the data points in piece-wise manner.
- The polynomial connecting (x_0, f_0) and (x_1, f_1) may be different from (x_1, f_1) and (x_2, f_2), etc.
- Splines are lower degree polynomials that connect sub-sets of data points.
- First-degree splines will be straight line between each points.
- While incorporating splines, we have to see that the function is consistent as per actual function (i.e. it should be differentiable, the slopes and curvatures are continuous).

- In linear splines, we miss the continuity of first and second derivatives (i.e. slopes and curvature).
- If we use second degree polynomials, we miss continuity of curvature.
- Therefore cubic splines (third degree spline) are the lower degree polynomials that can assure continuity of slope as well as curvature.
- To develop cubic spline
- Consider the (n+1) data points $(x_0,f_0),(x_1,f_1),\ldots,(x_n,f_n)$ in a data line below:



- There are (n+1) data points and n intervals.
- At each interval we want to develop cubic splines.

Intervals $j = 1, 2, 3, \ldots, n$

Cubic spline for the j^{th} interval is given as:

$$f_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$$

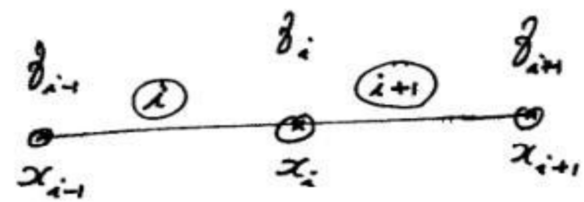
where for j^{th} interval, if $j = i, x_{i-1} \le x \le x_i$.

$\therefore$ We can write $f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$

- The conditions for cubic spline approximations

(i) Function values at data points are known

i.e. $f(x_0) = f_0$, $f(x_n) = f_n$ Also $f(x_i) = f_i$



$\rightarrow$ The data point x_i will be part of two splines $f_i(x)$ and $f_{i+1}(x)$.
So, we require $f_i(x_i) = f_{i+1}(x_i) = f_i$

(ii) The derivatives or slope should be same:

$f_i'(x) = b_i + 2c_ix + 3d_ix^2$

$f_{i+1}'(x) = b_{i+1} + 2c_{i+1}x + 3d_{i+1}x^2$

$f_i'(x_i) = f_{i+1}'(x_i)$

(iii) The second derivative $f_i''(x) = 2c_i + 6d_ix$

We also require $f_i''(x_i) = f_{i+1}''(x_i)$

(iv) The first spline i.e. $f_1(x)$ should pass through $x = x_{0.}$

The last spline i.e. $f_2(x)$ should pass through $x = x_n$.

(v) At $x = x_{0.}$; $f_1''(x_0)$ = specified $f''(x_0)$ (should be known)

At $x = x_{n.}$; $f_n''(x_n)$ = specified $f''(x_n)$) (should be known)

$\rightarrow f_i(x) = a_i + b_i x + 2c_i x^2 + 3d_i x^3$

$f_i'(x) = b_i + 2c_i x + 3d_i x^2$

$f_i''(x) = 2c_i + 6d_i x$

- This is a linear equation. We can represent it by Lagrange polynomial.

$$f_i''(x) = \frac{x - x_i}{x_{i-1} - x_i} f_i''(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f_i''(x_i)$$

As per the cubic spline criteria: $f_i''(x_{i-1}) = f_{i-1}''(x_{i-1}) = f_{i-1}''$

Similarly, $f_i''(x_i) = f_i''$

Also, $f_i'(x_{i-1}) = f_{i-1}'(x_{i-1}) = f_{i-1}'$ and $f_i'(x_i) = f_{i+1}'(x_{i-1}) = f_i'$

$$\therefore\ f_i'(x) = \int f_i''(x_i)dx = \frac{x^2/2 - xx_i}{x_{i-1} - x_i} f_{i-1}'' + \frac{x^2/2 - xx_{i-1}}{x_i - x_{i-1}} f_i'' + C$$

and

$$f_i(x) = \int f_i'(x_i)dx = \frac{x^3/6 - x^2/2\, x_i}{x_{i-1} - x_i} f_{i-1}'' + \frac{x^3/6 - x^2/2\, x_{i-1}}{x_i - x_{i-1}} f_i'' + Cx + D$$

- To find the integration parameters C and D, we have to utilize the known values at $x = x_{i-1}$ and $x = x_i$. i.e.

$$f_i'(x_i) = f_i', \; f_i'(x_{i-1}) = f_{i-1}', \; f_i(x_i) = f_i, \; f_i(x_{i-1}) = f_{i-1}$$

- However in actual, we have only the values

$f_0, f_1, f_2, \ldots, f_n$

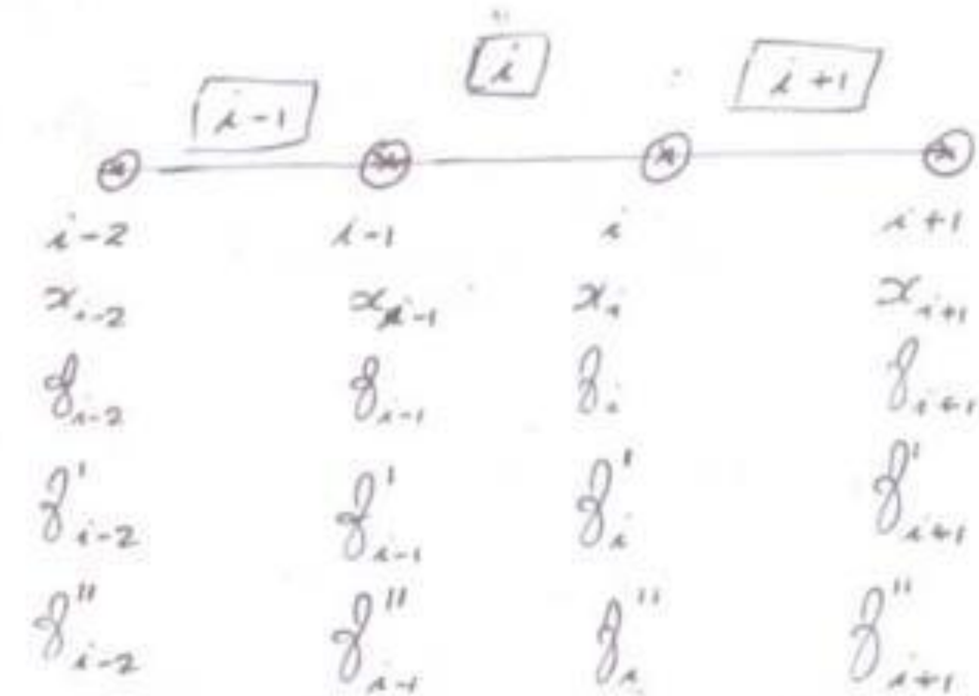
-> $(n+1)$ values

f_0' and f_n'

-> 2 values

f_0'' and f_n''

-> 2 values



- All other f_i' and f_i'' should be inferred from these values.

So for any interval,

$$f_i(x) = \frac{(x_i - x)^3}{6(x_i - x_{i-1})} f_{i-1}^{''} + \frac{(x - x_{i-1})^3}{6(x_i - x_{i-1})} f_i''$$

$$+(x_i - x)\left\{\frac{1}{(x_i - x_{i-1})} f_{i-1} - \frac{(x_i - x_{i-1})}{6} f_{i-1}^{''}\right\}$$

$$+(x - x_{i-1})\left\{\frac{f_i}{(x_i - x_{i-1})} - \frac{(x_i - x_{i-1})}{6} f_i''\right\} \quad \rightarrow (1)$$

- Differentiating this equation and applying at known points f_{i+1}', f_i' and f_{i+1}', finally we will get corresponding expressions.

- To find expressions for f_i'' at interval nodes $i=1, 2,3,\ldots,n-1$ obtained by differentiating eq. (1) once to get $f_i'(x)$ and again for $f_i''(x)$.
- The differentiated eq. $f_i'(x)$ is used for two known points f_{i-1}' and f_i'.
- Also differentiated eq. $f_{i+1}'(x)$ is used for f_i'.
- Similarly, $f_{i-1}'(x)$ is used for f_{i-1}'.
- Equating these equations, we get unknown in terms of f_{i-1}'', f_i'' and f_{i+1}''.

$$\rightarrow (x_i - x_{i-1}) f_{i-1}'' + 2(x_{i+1} - x_{i-1}) f_i'' + (x_{i+1} - x_i) f_{i+1}''$$

$$= 6\frac{f_{i+1} - f_i}{x_{i+1} - x_i} - 6\frac{f_i - f_{i-1}}{x_i - x_{i-1}} \qquad \rightarrow (2)$$

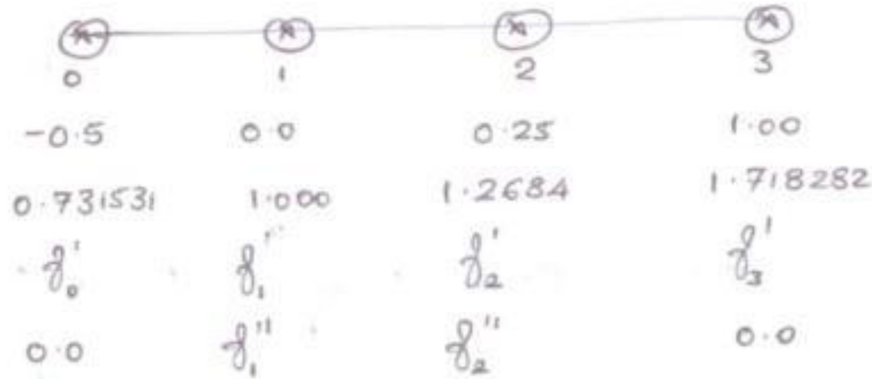
Eq. (2) is substituted in (n-1) interval nodes

to get systems of equations for unknown f_i'' $(i = 1, 2, 3, \ldots, n-1)$.

- Example (As adopted from the course text book Hoffman’s Numerical Methods)
- For the given data set

f

i	*x*	*f(x)*	*f”(x)*
0	-0.500	0.731531	0.0
1	0.000	1.00000	
2	0.250	1.26840	
3	1.000	1.718282	0.0



- Soln. There are 4 data points. Three intervals are there *i = 1,2,3*

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

$$= \frac{(x_i - x)^3}{6(x_i - x_{i-1})} f_{i-1}'' + \frac{(x - x_{i-1})^3}{6(x_i - x_{i-1})} f_i'' + (x_i - x)\left\{ \frac{1}{(x_i - x_{i-1})} f_{i-1} - \frac{(x_i - x_{i-1})}{6} f_{i-1}'' \right\}$$

$$+(x - x_{i-1})\left\{ \frac{f_i}{(x_i - x_{i-1})} - \frac{(x_i - x_{i-1})}{6} f_i'' \right\}$$

We have $f_0'' = f_3'' = 0.0$

$\therefore$ For $i = 1$ interval, we have $0.5 \le x \le 0$

i.e. $(x_1 - x_0)\, f_0'' + 2(x_2 - x_0)\, f_1'' + (x_2 - x_1)\, f_2''$

$$= 6\frac{f_2 - f_1}{x_2 - x_1} - 6\frac{f_1 - f_0}{x_1 - x_0}$$

i.e. $0.5 \times 0.0 + 2 \times 0.95 \times f_1'' + 0.25 \times f_2''$

$$= 6 \times \frac{1.2684 - 1}{0.25} - 6 \times \frac{1 - 0.731531}{0.5}$$

i.e. $1.50\, f_1'' + 0.25\, f_2'' = 3.219972$

Similarly, for interval $i = 2,\ 0 \le x \le 0.25$

We have,

$(x_2 - x_1)\, f_1'' + 2(x_3 - x_1)\, f_2'' + (x_3 - x_2)\, f_3''$

$$= 6\frac{f_3 - f_2}{x_3 - x_2} - 6\frac{f_2 - f_1}{x_2 - x_1}$$

i.e. $0.25 \times f_1'' + 2 \times 1 \times f_2'' + 0.75 \times 0.0 = -2.842544.$

Solving for f_1'' and f_2'', we get:

$f_1'' = 2.434240$ and $f_2'' = -1.725552$

$$\therefore\ f_1(x) = \frac{(x_1 - x)^3}{6(x_1 - x_0)} f_0'' + \frac{(x - x_0)^3}{6(x_1 - x_0)} f_1'' + \frac{(x_1 - x)}{(x_1 - x_0)} f_0$$

$$- \frac{(x_1 - x)(x_1 - x_0)}{6} f_0'' + \frac{(x - x_0)}{(x_1 - x_0)} f_1 - \frac{(x - x_0)(x_1 - x_0)}{6} f_1''$$

$$\text{i.e. } f_1(x) = \frac{(x + 0.5)^3}{6 \times 0.5} f_1'' + \frac{(0 - x)}{0.5} f_0 + \frac{(x + 0.5)}{0.5} f_1$$

$$- \frac{0.5 \times (x + 0.5)}{6} f_1''$$

Similarly obtain cubic polynomial for interval $i = 2$ and 3.