CE 601: Numerical Methods

Lecture 15

Difference Polynomials, Inverse Interpolation, Multi-Variate Approx.

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Difference Polynomials

- Rather than going for divided differences, we can simply tabulate the differences in function values and correspondingly develop difference polynomials

x_i	y_i			
x_0	y_0			
		$(y_1 - y_0)$		
x_1	y_1		$(y_2 - 2y_1 + y_0)$	$(y_3 - 3y_2 + 3y_1 - y_0)$
		$(y_2 - y_1)$		
x_2	y_2		$(y_3 - 2y_2 + y_1)$	
		$(y_3 - y_2)$		
x_3	y_3			
$\vdots$	$\vdots$			
x_n	y_n			

- You can form such difference tables. Based on the reference point, the type of differences are named.
  - Forward difference $\Delta f_i = f_{i+1} - f_i$
  - Backward difference $\nabla f_{i+1} = f_{i+1} - f_i$
  - Centered difference $\delta f_{i+1/2} = f_{i+1} - f_i$
- So you can have forward, backward ot centered difference table.
- In a given (n+1) data points available, you can only fit one unique n^{th} degree polynomial $P_n(x)$, irrespective of the methods (the polynomial has to pass through all points).

- For a given (n+1) data points available, you can only fit one unique n^{th} methods (the polynomial has to pass through all points).
- <u>Newton Forward Difference Polynomial</u>
- If the *x* value of a data set are uniformly spaced then let us define for the data set.

x_i	f_i
x_0	f_o
x_1	f_1
x_2	f_2
:	:
x_n	f_n

$$s = \frac{x - x_0}{\Delta x}$$

- This variable is called interpolating variable. i.e. *s* is linear in *'x'*.

Now the unique n^{th} degree polynomial can also be represented as,

$$P_n(x) = f_0 + s\Delta f_o + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots + \frac{s(s-1)(s-2)\cdots(s-(n-1))}{n!}\Delta^n f_0$$

This is Newton's forward difference polynomial.

Let us see the various situation:

When s = 0, i.e. $0 = \frac{x - x_0}{\Delta x} \Rightarrow x = x_0.$

Then $P_n(x_0) = f_0$ same as the polynomial.

When s = 1, i.e. $1 = \frac{x - x_0}{\Delta x} \Rightarrow x = x_0 + \Delta x = x_1$

$$\therefore P_n(x_1) = f_0 + \Delta f_0 + 0 = f_0 + \left(f_1 - f_0 \right) = f_1$$

In a similar note $P_n(x)$ passes through all the (n+1) data points given.

You can also represent this forward difference polynomial using binomial coefficient representation

i.e. $$\binom{s}{i} = \frac{s(s-1)(s-2)\cdots(s-(i-1))}{i!}$$

$$\therefore P_n(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \cdots$$

Example : For the earlier given data, develop Newton's forward difference polynomial and interpolate $f(t = 24 \text{ s})$.

t_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
10.00	50.00			
		90.00		
20.00	140.00		80.00	
		170.00		-20.00
30.00	310.00		60.00	
		230.00		
40.00	540.00			

We can start with various degrees of Newton's forward difference polynomial to interpolate for $f(t = 24 \text{ sec})$.

- Let us begin with first degree polynomial $P_1(t) \approx f(t)$. As our interpolation is required at $t = 24$ sec and also using forward difference, let the base point $t_i = 20.00$ sec.

$\therefore P_1(t) = f_i + s\Delta f_i = 140.00 + s \times 170.00 = 140 + 170s.$

For $t = 24$ sec, $s = (24 - 20)/10 = 0.4.$

$\therefore P_1(t = 24) = P_1(s = 0.4) = 140.00 + 170.00 \times 0.4 = 208$ m.

- Now let us use a second degree polynomial $P_2(t)$. Again keep the base point at $t = 20.0$ sec.

$$P_2(t) = P_2(s) = f_i + s\Delta f_i + \frac{s(s-1)}{2!}\Delta^2 f_i$$

$$P_2(s) = 140 + s \times 170 + \frac{s(s-1)}{2!} \times 60$$

As $s = 0.4, P_2(0.4) = 140 + 0.4 \times 170 + \dfrac{0.4(0.4-1)}{2!} \times 60 = 200.80$ m.

- The third degree polynomial will be unique for this case.

Therefore, it should begin with base point $t_0 = 10.0$ sec.

$$P_3(s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$= 50.00 + s \times 90 + \frac{s(s-1)}{2!} \times 80 + \frac{s(s-1)(s-2)}{3!} \times (-20.0)$$

For $t = 24$ seconds, $s = \dfrac{24-10}{10} = 1.4$

$$P_3(1.4) = 50.00 + 1.4 \times 90 + \frac{1.4 \times (1.4-1)}{2!} \times 80 + \frac{1.4 \times (1.4-1) \times (1.4-2)}{3!} \times (-20.0)$$

$$= 199.52 \text{ m.}$$

$\rightarrow$ So you can see each degree polynomial gives different results.

Newton Backward Difference Polynomial

As explained for Newton's forward difference polynomials, we can also use backward difference formulas to develop polynomials.

→ We can use backward differences

→ For a uniformly spaced data, if x_n is the base in an $(n+1)$ data $(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)$,

$\nabla f_n = f_n - f_{n-1}$, and define $s = \dfrac{x - x_n}{\Delta x}$.

We have the unique n^{th} degree polynomial in the power series as:

$$P_n(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \cdots + \frac{s(s+1)(s+2)\cdots(s+(n-1))}{n!}\nabla^n f_n$$

where $\nabla^{(i)} f_n \rightarrow$ shows the i^{th} difference.

The proof that this polynomial passes through all the data points can be given as below,

$\rightarrow$ Let $s = -1$; i.e. $s = -1 = \dfrac{x - x_n}{\Delta x} \Rightarrow x = x_n - \Delta x = x_{n-1}$.

$$P_n(x_{n-1}) = P_n(-1) = f_n + (-1)\left(f_n - f_{n-1}\right) + 0 = f_{n-1}.$$

$\rightarrow$ Again for $s = -2$; i.e. $s = -2 = \dfrac{x - x_n}{\Delta x} \Rightarrow x = x_n - 2\Delta x = x_{n-2}$.

$$P_n(x_{n-2}) = P_n(-2) = f_n + (-2)\left(f_n - f_{n-1}\right) + \frac{(-2)(-1)}{2!}\left[f_n - 2f_{n-1} + f_{n-2}\right] = f_{n-2},$$

So it's matching.

$\rightarrow$ You may work out the examples in this topic.

Inverse Interpolation

- For a given data set

x_i	f_i
x_0	f_0
x_1	f_1
x_2	f_2
:	:
x_n	f_n

- We generally consider ‘x’ as the independent quantity and ‘f’ as the dependent quantity.
- Using polynomial approximations, we also performed interpolations *f(x).*
- For the same data, one can also perform inverse interpolations. i.e. the function value ‘f’ may be available and we can determine the corresponding ‘x’ value., i.e. *x=f(x)*

- You can now use previously mentioned procedures to form polynomials $P_n(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$
  - Directly-fit polynomials
  - Lagrange polynomials
  - Newton’s polynomials, etc.

Multi-Variate Approximations

- We discussed till now in using polynomials as approximating functions for uni-variate conditions i.e. $f=f(x)$, etc.
- Again many scientific analysis may involve multi-variate relationships i.e. $Z=f(x,y)$, etc.
- To analyze such multi-variate problems, again we can use exact-fit polynomials.
- In the exact-fit procedure, two methods
  - Successive univariate polynomials approximation
  - Direct multi-variate polynomial approximation

Successive uni-variate polynomial approximation

- In some of your scientific or engineering problems you may come up with multi-variate data. E.g.

$y \backslash x$	x_1	x_2	x_3	$\cdots$	x_n
y_1	z_{11}	z_{12}	z_{13}	$\cdots$	z_{1n}
y_2	z_{21}	z_{22}	z_{23}	$\cdots$	z_{2n}
$\vdots$ y_3	z_{31}	z_{32}	z_{33}	$\cdots$	z_{3n}
$\vdots$	$\vdots$				
y_m	z_{m1}	z_{m2}	z_{m3}	$\cdots$	z_{mn}

- If such multi-variate data exist, then you may be interested to find a relation *Z = f(x,y)*.
- Again using polynomial approximation we can do interpolation, differentiation, integration etc.
- The method here is

  $*$ Fit uni-variate polynomials $Z_i(x) = Z(y_i, x)$ for each row

  i.e. $Z_i = a_0 + a_1 x + a_2 x^2 + \cdots$

  That is to find the respective coefficients for each line.

  $*$ If $x = x^*$ is the point where we need to do interploation, then obtain

  $$Z_1(x^*) = c_1$$
  $$Z_2(x^*) = c_2$$
  $$\vdots \qquad \vdots$$

  $Z_m(x^*) = c_m$. Using the m equations available.

* Now again a univariate polynomial can be obtained using the m-data points $c_1, c_2, \ldots, c_m$

i.e. $Z(y, x^*) = b_0 + b_1 y + b_2 y^2 + \cdots$

* After forming the polynomial by finding coefficients $b_0, b_1, \ldots$ we can interpolate for $Z(y^*, x^*)$

- Example: The enthalpy of a gaseous system *(H)* measured in kJ/kg by varying pressure (kPa) and temperature (^oC). The observations are recorded as such:

Pressure (kPa) \ Temp ^oC	427	538	649
7929	3211	3489	3755
8274	3205	3487	3753
8618	3199	3482	3751

- You are now requested to evaluate enthalpy at $T = 596\ ^oC$ and $P = 8446\ kPa$.

Solution

There are three different values of T and three different values of P.

Using successive univariate approximation:

$$H_1(7929,T) = a_{01} + a_{11}T + a_{21}T^2$$

$$H_2(8274,T) = a_{02} + a_{12}T + a_{22}T^2$$

$$H_3(8618,T) = a_{03} + a_{13}T + a_{23}T^2$$

You are having three different polynomials for H (i.e. H_1, H_2 and H_3). Find the coefficient for each.

For $H_1(7929,T) = a_{01} + a_{11}T + a_{21}T^2$

$$3211 = a_{01} + a_{11} \times 427 + a_{21} \times (427)^2$$

$$3489 = a_{01} + a_{11} \times 538 + a_{21} \times (538)^2$$

$$3755 = a_{01} + a_{11} \times 649 + a_{21} \times (649)^2$$

On solving you get, $a_{01} = 2.029706 \times 10^3, a_{11} = 2.974433, a_{21} = -0.00048697.$

$$\therefore H_1 = 2.029706\times10^3 + 2.974433T - 4.8697\times10^{-4}T^2$$

Similarly,

$$H_2 = H_2(8274,T) = 1.9710286\times10^3 + 3.167113T - 6.49298\times10^{-4}T^2$$

and

$$H_3 = H_3(8618,T) = 1.979827\times10^3 + 3.0978T - 5.681357\times10^{-4}T^2$$

Using these three polynomials we will evaluate enthalpy H at temperature $T = 596^\circ C$.

We get, $H_1(7929,596) = 3629.50$ kJ/kg,

$H_2(8274,596) = 3628$ kJ/kg,

$H_3(8618,596) = 3624$ kJ/kg.

Using these three evaluated values another polynomial of second degree is constructed as such: $H(P,596) = b_0 + b_1P + b_2P^2$

where b_0, b_1 and b_2 coefficients are evaluated.

$$3629.50 = b_0 + 7929\times b_1 + (7929)^2 b_2$$

$$3628 = b_0 + 8274\times b_1 + (8274)^2 b_2$$

$$3624 = b_0 + 8618\times b_1 + (8618)^2 b_2$$

Solving the above you get: $b_0 = 2970.786, b_1 = 0.166856, b_2 = -1.056616\times10^{-5}$.

To interpolate, $H(8446,596) = 3626.31$ kJ/kg.