CE 601: Numerical Methods

Lecture 14

Polynomial Approximations (contd..)

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Polynomial Approximations

- Polynomials can be used (and are most popular) as approximating functions to a given data set.
- There are
  - → Exactly-fit polynomials
  - → Approximately fitting polynomials
- In the exactly-fit polynomials, based on the desired number of points we consider, exactly-fit polynomials can be developed.
- For a total of ($n+1$) data points, one can develop exactly fit $P_1(x)$, $P_2(x)$, … ,$P_n(x)$ polynomials.
- However for a given ($n+1$) data, the polynomial $P_n(x)$ will always be unique.

Direct-Fit Polynomials

- In the given *(n+1)* data set $(x_0, f_0), (x_1, f_1),.., (x_n, f_n)$.
- As discussed there can be only one unique n^{th} degree polynomial.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Substitute the polynmial at each data point.

$$\begin{aligned}
f_0 &= a_0 + a_1 x_0 + a_2 x_0^{\,2} + \cdots + a_n x_0^{\,n} \\
f_1 &= a_0 + a_1 x_1 + a_2 x_1^{\,2} + \cdots + a_n x_1^{\,n} \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
f_n &= a_0 + a_1 x_n + a_2 x_n^{\,2} + \cdots + a_n x_n^{\,n}
\end{aligned}$$

- You will get *(n+1)* equations with *(n+1)* unknowns $a_0, a_1, a_2, \ldots, a_n$
- This forms a system of linear equations. You solve for $a_0, a_1, a_2, \ldots, a_n$ and you will get the n^{th} degree polynomial.

Example

x	f
3.35	**0.298509**
3.40	**0.294118**
3.50	**0.285714**
3.60	**0.277778**

- Let us fit a polynomial approximation to relate '*f*' with '*x*'.
- The unique polynomial can be say, $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

$$\text{Now, } f_0 = 0.298509 = a_0 + 3.35a_1 + (3.35)^2 a_2 + (3.35)^3 a_3$$

$$f_1 = 0.294118 = a_0 + 3.40a_1 + (3.40)^2 a_2 + (3.40)^3 a_3$$

$$f_2 = 0.285714 = a_0 + 3.50a_1 + (3.50)^2 a_2 + (3.50)^3 a_3$$

$$f_3 = 0.277778 = a_0 + 3.60a_1 + (3.60)^2 a_2 + (3.60)^3 a_3$$

You are getting a system of linear equations,

$$\begin{pmatrix} 1 & 3.35 & 11.2225 & 37.5954 \\ 1 & 3.40 & 11.5600 & 39.3040 \\ 1 & 3.50 & 12.2500 & 42.8750 \\ 1 & 3.60 & 12.9600 & 46.6560 \end{pmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0.298509 \\ 0.294118 \\ 0.285714 \\ 0.277778 \end{Bmatrix}$$

- Solving this, you will get a unique polynomial.
- You can then subsequently use it for:
  - ❑ Interpolation
  - ❑ Differentiation
  - ❑ Integration

Lagrange Polynomials

- The direct fit polynomial tries to solve a system of linear equations. The number of operations involved may be huge and computationally intensive. It may also be time consuming.
- There are other ways to determine the polynomials that pass through all the desired points.
- One such procedure is Lagrange's polynomials.
- If there are two data points set (x_0, f_0) and (x_1, f_1), then we can fit the unique first degree polynomial $P_1(x)$.

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f_0 + \frac{(x - x_0)}{(x_1 - x_0)} f_1$$

If three data points: $(x_0, f_0), (x_1, f_1)$ and (x_2, f_2), then:

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

An n^{th} degree Lagrange polynomial will be:

$$P_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f_1$$

$$\vdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f_n$$

- Lagrange polynomials can be used for equally spaced and unequally spaced data.

Example (same example)

- Say if you want to develop a $P_2(x)$ that passes through (x_0, f_0), (x_1, f_1) and (x_2, f_2).

$$P_2(x) = \frac{(x-3.40)(x-3.50)}{(3.35-3.40)(3.35-3.50)} \times 0.298509 + \frac{(x-3.35)(x-3.50)}{(3.40-3.35)(3.40-3.50)} \times 0.294118$$
$$+ \frac{(x-3.35)(x-3.40)}{(3.50-3.35)(3.50-3.40)} \times 0.285714$$

- Similarly $P_3(x)$ can be developed and it will be unique.
- Note:

1) The direct-fit polynomials are useful mostly when data are uniformly spaced.
2) In Lagrange method, we need to reevaluate the entire calculations again if we need to change the polynomial. (i.e. from $P_2(x)$ to $P_3(x)$).

Divided Difference Polynomials

- Divided difference

  -> ratio of difference in the function values at two points and the difference in the corresponding independent variables.
- Consider the (n+1) data points *(x_i, f_i)*.

x_i	f_i
x_0	f_0
x_1	f_1
x_2	f_2
:	:
x_n	f_n

- We can evaluate divided difference for any data point.
- First Divided Difference

At any data point $'i'$

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

$$\therefore \text{First divided difference at } i = 0,\ f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

The second divided difference at any data no. 'i',

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$= \frac{\dfrac{(f_{i+2} - f_{i+1})}{(x_{i+2} - x_{i+1})} - \dfrac{(f_{i+1} - f_i)}{(x_{i+1} - x_i)}}{x_{i+2} - x_i}$$

- We can define third divided diference and so on.
- The n^{th} divided difference for data point $i = 0$,

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$$

Rather than writing $f[x_0, x_1]$ or $f[x_i, x_{i+1}]$, symbolically

we will write as $f_0^{(1)}, f_i^{(1)}$.

$f[x_i, x_{i+1}, x_{i+2}] \rightarrow f_i^{(2)}$

$f[x_i, x_{i+1}, \cdots, x_{i+n}] \rightarrow f_i^{(n)}$ in the general form.

- The Divided difference table will be:

x_i	f_i	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
x_0	f_0			
		$f_0^{(1)}$		
x_1	f_1		$f_0^{(2)}$	
		$f_1^{(1)}$		$f_0^{(3)}$
x_2	f_2		$f_1^{(2)}$	
		$f_2^{(1)}$		
x_3	f_3			

- You can easily evaluate divided differences for any data.

- Q. What is the purpose of divided difference ?
- Divided differences are used to develop polynomials.
- A n^{th} degree divided difference polynomial for (n+1) data.

$$P_n(x) = f_i + (x - x_0) f_i^{(1)} + (x - x_0)(x - x_1) f_i^{(2)}$$

$$+ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f_i^{(n)}$$

- This is actually defining a power series for $P_n(x)$ with its coefficient being the divided differences $f_i^{(n)}$.
- This *nth* degree polynomial will definitely pass through (x_i, f_i).

Example

t_i (x)	f_i	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0.000	0.00	5.000	0.250	0.00	0.000	0.000
10.000	50.00	10.000	0.250	0.00	0.000	
20.000	150.00	15.000	0.250	0.000		
30.00	300.00	20.00	0.250			
40.00	500.00	25.000				
50.00	750.00					

- For the above example table of divided difference, we can develop a unique *5th* degree polynomial with the beginning data point *i=0.*

$$P_5(x) = f_0 + (t-t_0)\,f_i^{(1)} + (t-t_0)(t-t_1)\,f_0^{(2)}$$
$$+(t-t_0)(t-t_1)(t-t_2)\,f_0^{(3)} + (t-t_0)(t-t_1)(t-t_2)(t-t_3)\,f_0^{(4)}$$
$$+(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)\,f_0^{(5)}$$
$$= 0.00 + 5t + t(t-10)\times 0.25 + 0.0 + 0.0$$

Difference Polynomials

- Rather than going for divided differences, we can simply tabulate the differences in function values and correspondingly develop difference polynomials

x_i	y_i			
x_0	y_0			
		$(y_1 - y_0)$		
x_1	y_1		$(y_2 - 2y_1 + y_0)$	$(y_3 - 3y_2 + 3y_1 - y_0)$
		$(y_2 - y_1)$		
x_2	y_2		$(y_3 - 2y_2 + y_1)$	
		$(y_3 - y_2)$		
x_3	y_3			
$\vdots$	$\vdots$			
x_n	y_n			

- You can form such difference tables. Based on the reference point, the type of differences are named.
  - Forward difference $\Delta f_i = f_{i+1} - f_i$
  - Backward difference $\nabla f_{i+1} = f_{i+1} - f_i$
  - Centered difference $\delta f_{i+1/2} = f_{i+1} - f_i$
- So you can have forward, backward or centered difference table.
- In a given (n+1) data points available, you can only fit one unique *n^{th}* degree polynomial *$P_n(x)$,* irrespective of the methods (the polynomial has to pass through all points).
- For a given (n+1) data points available, you can only fit one unique *n^{th}* methods (the polynomial has to pass through all points).