CE 601: Numerical Methods

# Lecture 12

# Solutions of Polynomials & System of Non-Linear Equations

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# Solutions of Polynomials

- While using Newton's method, after finding first root $x=\alpha$, then we need to deflate the polynomial to find subsequent roots.
- Q. What happens to Newton's method or how Newton's method behave if there are multiple roots for a polynomial (or function)
- Newton's method of Multiple Roots

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + .... + a_n x^n$$

Let $x=\alpha$ be a root that occurs 'm' times.

$$x_{i+1} = x_i - f(x_i)/f'(x_i)$$

→ The Newton's method earlier was told to have quadratic convergence.

→ But in case of multiple roots the Newton's method reduces to linear convergence.

e.g. $P_3(x) = x^3 - 1.2502x^2 - 1.5625x + 1.95344 = 0$

Now $f'(x) = 3x^2 - 2.5004x - 1.5625$

| i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | Error |
|---|---|---|---|---|
| 0 | 1.50000 | 0.17174 | 1.43690 | 0.17174 |
| 1 | 1.38048 | 0.04472 | 0.70292 | 0.04472 |
| 2 | 1.31686 | 0.01144 | 0.34718 | 0.01144 |
| 3 | 1.28391 | 0.00290 | 0.17249 | 0.00290 |
| 4 | 1.26710 | 0.00073 | 0.08587 | 0.00073 |

→ The solution is not yet converged. It is taking more time, even after that $error < 10^{-2}$ since iteration 3.

→ This is because the Newton's method above is not quadratically converging.

→ To form a quadratic function, you require at least three known points.

- For $P_n(x) = a_0 + a_1 x + a_2 x^2 + .... + a_n x^n = f(x)$

  If $x=\alpha$ is a root that appears 'm' times, then we can deflate the polynomial as

$$P_n(x) = (x- \alpha)^m A(x) = f(x)$$

where $x=\alpha$ is not a root of $A(x)$.

# Newton's method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow x_{i+1} = x_i - \frac{A(x_i)(x_i-\alpha)^m}{\left[mA(x_i)(x_i-\alpha)^{m-1} + A'(x_i)(x_i-\alpha)^m\right]}$$

$$\Rightarrow x_{i+1} - \alpha = x_i - \alpha - \frac{A(x_i)(x_i-\alpha)^m}{\left[mA(x_i)(x_i-\alpha)^{m-1} + A'(x_i)(x_i-\alpha)^m\right]}$$

$$\Rightarrow x_{i+1} - \alpha = \left(x_i - \alpha\right)\left[1 - \frac{1}{m + \frac{A'(x_i)(x_i-\alpha)}{A(x_i)}}\right]$$

- When '*i*' is sufficiently large and $x_i \longrightarrow \alpha$, then

$$e_{i+1} = e_i (1-1/m)$$

that is the rate of convergence is linear.

- To overcome this situation if you know the multiplicity '*m*' of a root of the $P_n(x)=f(x)$, then we need to develop another function.

  $g(x) = [f(x)]^{1/m}$ such that $x=\alpha$ is also the root of the function $g(x)=1/[f(x)]^m$.

$$\therefore x_{i+1} = x_i - \frac{[f(x_i)]^{1/m}}{\frac{d}{dx}\left\{[f(x_i)]^{1/m}\right\}} = x_i - \frac{[f(x_i)]^{1/m}}{\frac{1}{m}[f(x_i)]^{1/m-1} f'(x_i)}$$

$$\Rightarrow x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

- <u>Same example</u>

$P_3(x) = x^3-1.2502x^2-1.5625x+1.95344 = 0 = f(x)$

$f'(x) = 3x^2-2.5004x-1.5625$

If we know that one root exists two times, then

$$x_{i+1} = x_i - 2\frac{f(x_i)}{f'(x_i)}$$

| i | $x_i$ | $f(x_i)$ | $f'(x_i)$ | Error |
|---|---|---|---|---|
| **0** | **1.50000** | **0.17174** | **1.43690** | **0.17174** |
| **1** | **1.26096** | **0.00030** | **0.05466** | **0.00030** |
| **2** | **1.24998** | **$2.5 \times 10^{-6}$** | **-** | **$2.5 \times 10^{-6}$** |

- If the multiplicity of the root is unknown, then how to use Newton's method?
- Assign a new function *u(x) = f(x)/f'(x)*

If *x=α* appears 'm' times, then

$$f(x) = (x-\alpha)^m A(x); \quad A(x) \neq 0.$$

$$\therefore u(x) = \frac{(x-\alpha)^m A(x)}{m(x-\alpha)^{m-1} A(x) + (x-\alpha)^m A'(x)}$$

$$\text{i.e., } u(x) = \frac{(x-\alpha) A(x)}{mA(x) + (x-\alpha) A'(x)}$$

- If *f(x) = 0* is having *x=α* as a root, then *x=α* is also root of *u(x).*

$$\therefore x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$u'(x) = \frac{f'(x)f'(x) - f(x)f''(x)}{\left[f'(x)\right]^2}$$

$$\therefore x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{\left[f'(x_i)\right]^2 - f(x_i)f''(x_i)}$$

- You can see that value of ‘m’ is not used in the above expression.

# System of Non-linear Equations

- As discussed in earlier chapters – many of the engineering and scientific problems can be mathematically described.
- During these descriptions we may arrive at system of equation
  - The system of linear equations already seen.
  - The system of non-linear equations also appear for many phenomena.

- How to solve such systems?

i.e. $f(x_1, x_2, x_3, \ldots, x_n) = 0$

$$g(x_1, x_2, x_3, \ldots, x_n) = 0$$

$$\vdots$$

The solution is $\{x\} = \{x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n\}^T$

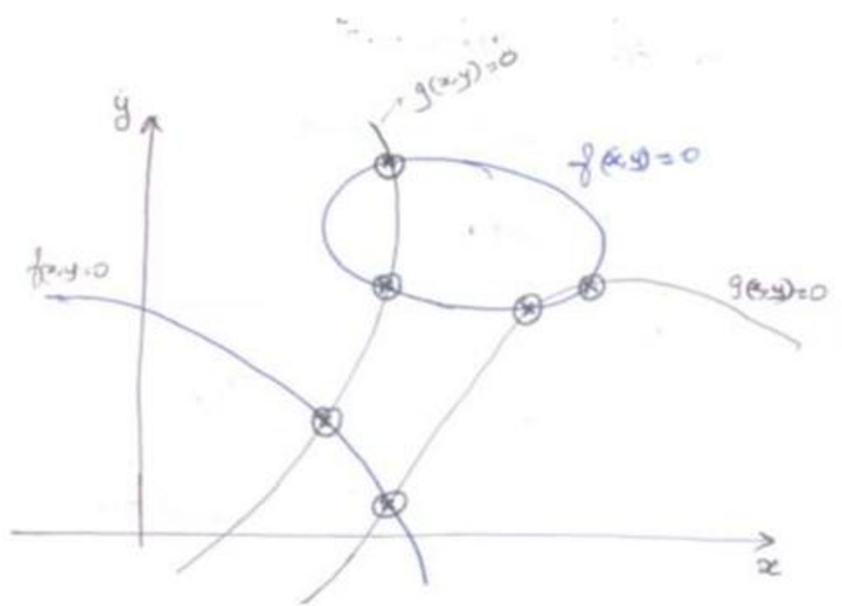
- A simple 2-D system

$$f(x,y) = 0$$

$$g(x,y) = 0$$

  - We need to find the solution $\{x\} = \{x \;\; y\}^T$.
  - In layman's method we can find the solution.

- Plot the curves *f(x,y)=0* in the x-y plane. These may be more than one curve.
- Plot the curves having *g(x,y)=0* in the same plane.
- The intersection points of *f(x,y)=0* and *g(x,y)* curve gives the solution.
- Due to non-linearity these may be many roots for a system.



# Iterative Method to solve non-linear systems

- The iterative methods like
  - Fixed point iteration
  - Newton's method
  - Secant method
  - Muller's method, etc.

can be used to solve the system of non-linear equations.

e.g. for a system of two linear equations

$$f(x,y) = 0$$
$$g(x,y) = 0$$

- Start with some initial guess $(x^{(0)},y^{(0)})$.
- Iterate and at nay iteration '*s*' you have $(x^{(s)},y^{(s)})$.
- Continue iteration till convergence.
- Now at any iteration 's', $(x^{(s)},y^{(s)})$ is known, we can use Taylor's series for *f(x,y)*.

$$f(x,y) = f(x^{(s)},y^{(s)}) + \left.\frac{\partial f}{\partial x}\right|_{(x^{(s)},y^{(s)})}(x - x^{(s)}) + \left.\frac{\partial f}{\partial y}\right|_{(x^{(s)},y^{(s)})}(y - y^{(s)}) + \dots$$

Similarly,

$$g(x,y) = g(x^{(s)},y^{(s)}) + \left.\frac{\partial g}{\partial x}\right|_{(x^{(s)},y^{(s)})}(x - x^{(s)}) + \left.\frac{\partial g}{\partial y}\right|_{(x^{(s)},y^{(s)})}(y - y^{(s)}) + \dots$$

Truncating after first order terms, we have

$$f(x,y) = f^{(s)} + \left.\frac{\partial f}{\partial x}\right|^{(s)} (x - x^{(s)}) + \left.\frac{\partial f}{\partial y}\right|^{(s)} (y - y^{(s)})$$

$$g(x,y) = g^{(s)} + \left.\frac{\partial g}{\partial x}\right|^{(s)} (x - x^{(s)}) + \left.\frac{\partial g}{\partial y}\right|^{(s)} (y - y^{(s)})$$

where $f^{(s)} = f(x^{(s)}, y^{(s)})$; $\left.\frac{\partial f}{\partial x}\right|^{(s)} = \left.\frac{\partial f}{\partial x}\right|_{(x^{(s)},\, y^{(s)})}$ and same for other terms.

$\therefore$ We will get,

$$\left.\frac{\partial f}{\partial x}\right|^{(s)} (x - x^{(s)}) + \left.\frac{\partial f}{\partial y}\right|^{(s)} (y - y^{(s)}) = -f^{(s)}$$

$$\left.\frac{\partial g}{\partial x}\right|^{(s)} (x - x^{(s)}) + \left.\frac{\partial g}{\partial y}\right|^{(s)} (y - y^{(s)}) = -g^{(s)}$$

Let us define $\Delta x^{(s)} = x - x^{(s)}$ and $\Delta y^{(s)} = y - y^{(s)}$

Therefore,

$$\left.\frac{\partial f}{\partial x}\right|^{(s)} \Delta x^{(s)} + \left.\frac{\partial f}{\partial y}\right|^{(s)} \Delta y^{(s)} = -f^{(s)}$$

$$\left.\frac{\partial g}{\partial x}\right|^{(s)} \Delta x^{(s)} + \left.\frac{\partial g}{\partial y}\right|^{(s)} \Delta y^{(s)} = -g^{(s)}$$

or, $$\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}^{(s)} \begin{Bmatrix} \Delta x \\ \Delta y \end{Bmatrix}^{(s)} = -\begin{Bmatrix} f \\ g \end{Bmatrix}^{(s)}$$

i.e.

$$[J]^{(s)}\{\Delta x\}^{(s)} = -\{f\}^{(s)}$$

Therefore, $\{\Delta x\}^{(s)} = -\left([J]^{(s)}\right)^{-1}\{f\}^{(s)}$

And; $\{x\}^{(s+1)} = \{x\}^{(s)} + \{\Delta x\}^{(s)}$

$$\{x\}^{(s+1)} = \{x\}^{(s)} - \left([J]^{(s)}\right)^{-1}\{f\}^{(s)}$$