CE 601: Numerical Methods

Lecture 11

Secant Method and Muller's Method

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- <u>Convergence criteria in N-R method</u>

- $e_{i+1} = x_{i+1} – \alpha$

  As $x_{i+1} = x_i - f(x_i)/ f'(x_i) = g(x_i)$ (say)

- For convergence we now require $|g'(\zeta)| < 1$

  $g(x) = x - f(x)/ f'(x)$

  So, $g'(x) = f(x) f''(x)/ (f'(x))^2$

- At exact convergence i.e. at $x = \alpha$, we have

  $x_{i+1} = x_i - f(x_i)/ f'(x_i)$

  $\Rightarrow x_{i+1} - \alpha = x_i - \alpha - f(x_i)/ f'(x_i)$

  $\Rightarrow e_{i+1} = e_i - f(x_i)/ f'(x_i)$

- Again using Taylor's series expression for $f(\alpha)$ w.r.t. x_i.

$$f(\alpha) = f(x_i) + f'(x_i)(\alpha - x_i) + (½) f''(\zeta) (\alpha - x_i)^2 = 0, x_i \le \zeta \le \alpha$$

i.e. $f(x_i) = e_i f'(x_i) - (½) f''(\zeta) e_i^2$

So, $e_{i+1} = e_i - f(x_i)/ f'(x_i)$ becomes

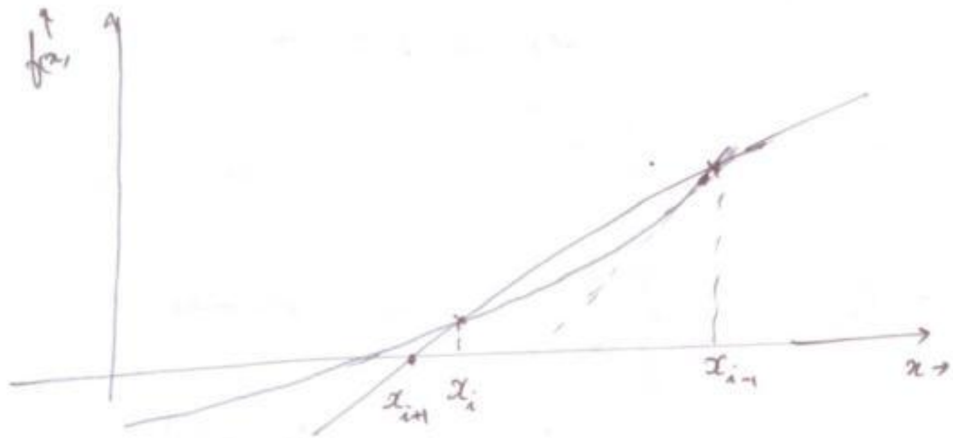
$\Rightarrow e_{i+1} = e_i - (e_i f'(x_i) - (½) f''(\zeta) e_i^2)/ f'(x_i)$

$\Rightarrow e_{i+1} = (½) f''(\zeta) e_i^2 / f'(x_i)$

- You can see that error reduces quadratically at every iteration. Therefore it is faster.
- Q. What happens if $f'(x) = 0$ while using N-R method?
- Then, Newton-Raphson method will not be suitable to obtain solutions.
- In that case, we need to go for another method called Secant Method.

- ## Secant method

- Instead of a tangential line, we will be drawing a secant line passing trough $(x_i, f(x_i))$.



- Here $x_{i+1} = x_i - f(x_i)/ g'(x_i)$. $g'(x_i)$ -> slope of the secant line. Now $g'(x_i) \neq f'(x_i)$

$$g'(x_i) = (g(x_i)-g(x_{i-1}))/(x_i-x_{i-1}) = (f(x_i)-f(x_{i-1}))/(x_i-x_{i-1})$$

$$g'(x_{i+1}) = (g(x_{i+1})-g(x_i))/(x_{i+1}-x_i) = -f(x_i) /(x_{i+1}-x_i)$$

Equating $(f(x_i)-f(x_{i-1}))/(x_i-x_{i-1}) = -f(x_i) /(x_{i+1}-x_i)$

Or, $\boxed{x_{i+1}= x_i - f(x_i)\,(x_i-x_{i-1})/ (f(x_i)-f(x_{i-1}))}$

- In the secant method, the improved value of *x* will be

$$x_{i+1} = x_i - f(x_i)\,(x_i - x_{i-1}) / (f(x_i) - f(x_{i-1}))$$

- Recall this particular expression

  → Compare with Regula-Falasi method

  → Identify what are the differences?

  → The students are requested to work out the examples on this topic

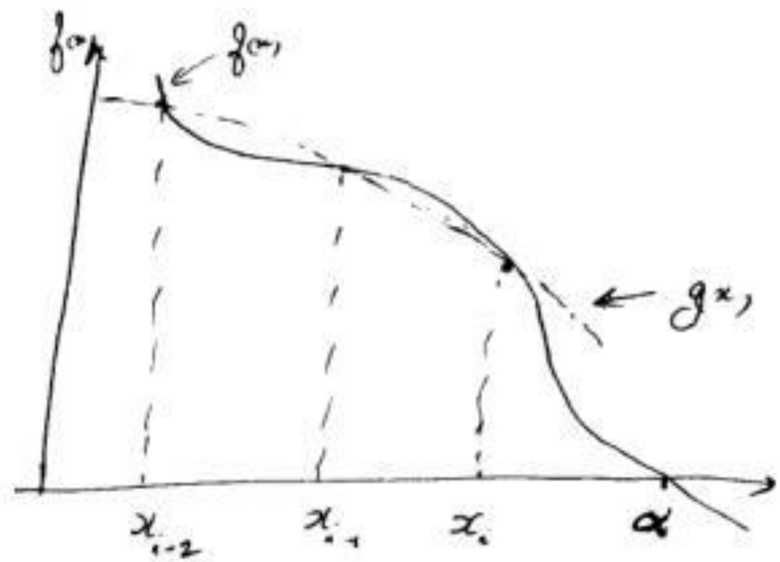
- <u>Muller's method</u>

→ In Newton's method and Secant method we approximated the non-linear function $f(x) \approx g(x)$ a straight line. A straight line is a first degree equation.

→ On a similar note we can approximate $f(x) \approx g(x)$, where $g(x)$ will be quadratic.

→ To form a quadratic function, you require at least three known points.

→ At any (i+1)th iteration you need to utilize x_{i-1}, x_{i-2}, and x_i.



→ Here $f(x) \approx g(x) = a\,(x - x_i)^2 + b(x-x_i) + c$

→ The known points are: $(x_i, f(x_i))$, $(x_{i-1}, f(x_{i-1}))$ and $(x_{i-2}, f(x_{i-2}))$. Also these three points the quadratic function $g(x)$ also passes

i.e. $g(x_i) = a \times 0 + b \times 0 + c = c = f(x_i)$

$g(x_{i-1}) = a\,(x_{i-1} - x_i)^2 + b\,(x_{i-1}-x_i) + c = f(x_{i-1})$

$g(x_{i-2}) = a\,(x_{i-2} - x_i)^2 + b\,(x_{i-2}-x_i) + c = f(x_{i-2})$

From these three relationships, we get $c = f(x_i)$.

Again symbolical term $\Delta x_1^{(i)}= x_{i-1} - x_i$, $\Delta x_2^{(i)}= x_{i-2} - x_i$,

$\delta f_1^{(i)}= = f(x_{i-1}) - f(x_i)$, $\delta f_2^{(i)}= = f(x_{i-2}) - f(x_i)$,.

i.e. $f(x_{i-1}) - f(x_i) = a\,(x_{i-1} - x_i)^2 + b\,(x_{i-1}-x_i)$

or, $\delta f_1^{(i)}= a\,(\Delta x_1^{(i)})^2 + b\,\Delta x_1^{(i)}$...... (1)

and $\delta f_2^{(i)}= a\,(\Delta x_2^{(i)})^2 + b\,\Delta x_2^{(i)}$ ...... (2)

→ Similarly we get,

$a = (\delta f_1 \ \Delta x_2 \ - \delta f_2 \Delta x_1) / (\Delta x_1 \Delta x_2 (\Delta x_1 - \Delta x_2))$

and $b = (\delta f_2 \Delta x_1^2 - \delta f_1 \Delta x_2^2) / (\Delta x_1 \Delta x_2 (\Delta x_1 - \Delta x_2))$

So, $g(x) = a (x – x_i)^2 + b(x-x_i) + c$

At x_{i+1}, we are expecting convergence.

i.e. $g(x_{i+1}) = 0 = a (x_{i+1} – x_i)^2 + b (x_{i+1}-x_i) + c$

i.e. $(x_{i+1}-x_i) [a (x_{i+1} – x_i) + b] = -c$

We have, $x_{i+1} – x_i = (-b \quad \surd(b^2 – 4ac)) / 2a$

We can wait, $(-b \quad \surd(b^2 – 4ac)) / 2a$

$= (-b \quad \surd(b^2 – 4ac)) (-b \quad \surd(b^2 – 4ac)) / 2a (-b \quad \surd(b^2 – 4ac))$

$= - 2c / (-b \quad \surd(b^2 – 4ac))$

So. $x_{i+1} = x_i = - 2c / (-b \quad \surd(b^2 – 4ac))$

where $c = f(x_i)$, $a = (\delta f_1^{(i)} \Delta x_2^{(i)} - \delta f_2^{(i)} \Delta x_1^{(i)}) / (\Delta x_1^{(i)} \Delta x_2^{(i)} (\Delta x_1^{(i)} - \Delta x_2^{(i)}))$

and $b = (\delta f_2^{(i)} (\Delta x_1^{(i)})^2 - \delta f_1^{(i)} (\Delta x_2^{(i)})^2) / (\Delta x_1^{(i)} \Delta x_2^{(i)} (\Delta x_1^{(i)} - \Delta x_2^{(i)}))$

→ Convergence criteria: $| x_{i+1} – x_i| \le \varepsilon$ or, $| f(x_{i+1}) | \le \varepsilon$.

- <u>Solutions of Polynomials</u>
- You have linear and non-linear polynomials

  $P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ is an n^{th} degree polynomial.

→ For any n^{th} degree polynomial, you can have n roots.

  → Roots may be real or complex.

  →The roots can be single or multiple.

→ Using open domain methods like – Newton, Secant, Muller etc we can find roots.

→ Recall that Newton's method converge quadratic ally.

$$e_{i+1} = (½) f''(\zeta)\, e_i^2 / f'(\zeta))$$

→ Let we use Newton's method to find the roots of the polynomial.

- <u>Example</u>: $f(x) = x^3 - 3x^2 + 4x - 2 = P_3(x)$
- $f(x) = f(x) = x^3 - 3x^2 + 4x - 2$ and $f'(x) = P_3(x) = 3x^2 - 6x + 4$
- $x_{i+1} = x_i - f(x_i)/f'(x_i)$
- Let us start with $x_0 = 2.0000.$

i	X_i	$f(x_i)$	$f'(x_i)$	Error
0	2.00000	2.00000	4.00000	2.00000
1	1.50000	0.62500	1.75000	0.62500
2	1.14286	0.14578	1.06123	0.14578
3	1.00549	0.00549	1.00009	0.00549
4	1.00000	0.00000		0.00000

- The solution is $\alpha = 1.0000.$ This is our root. There are still two roots. How will we find them?

- Use polynomial Deflation.
- Any nth degree polynomial

$$P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_n x^n$$

say if it is having a root $x = \alpha$, then we can write

$P_n(x) = (x - \alpha)\, Q_{n-1}(x)$

where $Q_{n-1}(x)$ is an (n-1)th degree polynomial.

$Q_{n-1}(x) = = b_1 + b_2x + b_3x^2 + ... + b_n x^{n-1}$

you can see that

$b_n = a_n$

$b_i = a_i + \alpha\, b_{i+1}\,;\ i = n\text{-}1,\ n\text{-}2,\ \ldots,\ 1$

So, in the given problem. $P_3(x) = x^3 - 3x^2 + 4x - 2$

i.e. $a_0 = -2$, $a_1 = 4$, $a_2 = -3$ *and* $a_3 = 1$.

$\alpha = 1.00000$ is an root.

So, $P_3(x) = (x - 1.00000)Q_2(x)$

Where $Q_2(x) = = b_1 + b_2x + b_3x^2$

Now, $b_3 = a_3 = 1.00000$

$$b_2 = a_2 + \alpha b_3 = -3 + 1.00 X 1.00 = -2.00000$$

$$b_1 = a_1 + \alpha b_2 = 4 + 1.00 X -2.00 = 2.00000$$

So, $Q_2(x) = 2 -2x + x^2$

→ The second degree polynomial can be again subjected to Newton's method to obtain the solution.