CE 601: Numerical Methods

Lecture 10

Open Domain Methods For Solutions of Non-Linear Equations

Course Coordinator:
Dr. Suresh A. Kartha,
Associate Professor,
Department of Civil Engineering,
IIT Guwahati.

- We have Seen the two closed domain methods - i) Method of bisection and ii) Method of false position.
- For any non-linear equation $f(x) = 0$, if say ‘α ‘is one exact solution, and if by using closed domain method if the iteration begins with x_o.

  The iterative sequence improves $x_1 \rightarrow x_1 \rightarrow x_2 \ldots$

  This will or should converge to α.

  So, Error in any iteration $e_i = \alpha - x_i$ and

  $$\lim_{i \to \infty} \frac{|e_{i+1}|}{|e_i|^p} = C$$

  and p is order of convergence,

  C is asymptotic error constant.

  If $p = 1$, linear convergence, $p = 2$, quadratic order of convergence, etc.
- For method of bisection, $p = 1$ and $C = ½$.
- Method of false position, $p = 1$, C depends on the slope and curvature of the function.

- From these two methods you can see that closed domain method guarantees convergence. However order of convergence was linear.
- Alternative to overcome the slowness is to use open domain methods.
- Such methods require only one or two points to start the iteration.
- <u>Open Domain Methods:</u>

o The roots are not bracketed as like in closed domain methods.

- ❖ Fixed-point iteration
- ❖ Newton's method
- ❖ Secant method
- ❖ Muller's method etc

- <u>Fixed Point Iteration</u>
- We have seen the non-linear equation is $f(x) = 0$.
- All equations of the form $f(x) = 0$ can be written as $x = g(x)$.
- E.g. in the example problem $f(x) = 2 - x^2 = 0$

  i.e. $0 = 2/x - x$

  or, $x = 2/x \quad (x = g(x))$
- If β is the root of $f(x) = 0$, then according to this form $\beta = g(\beta)$ i.e. the term function suggest about mapping of the value of x to a new location. If the mapped value and value of 'x', then it becomes a fixed point.

- Now in this method the process is
  - To solve $f(x) = 0$.
  - Rearrange $f(x)$ in such a way that $x = g(x)$
    - → Provide initial guess for x say x_i
    - → Evaluate $g(x_i)$
    - → If not equal then, $x_{i+1} = g(x_i)$
    - → Evaluate $g(x_{i+1})$
    - → Continue till some tolerance ε i.e., $|x_{i+1} - x_i| \leq \varepsilon$

- <u>Example:</u> Solve $f(x) = f(x) = x^2 -10x + 23.$
  - $f(x) = x^2 -10x + 23.$

    Now develop $x = g(x)$ form,

    $$x = 10 - (23/x) = g(x)$$

    Start initially say $x_o = 3.00000$ and

    suggest tolerance $\varepsilon = 1 \text{ X } 10^{-4}$ *for* $|x_{i+1} - x_i|$.

i	x_i	$g(x_i)$	$\|x_{i+1} - x_i\| \leq \varepsilon$
0	**3.00000**	**3.28571**	
1	**3.28571**	**3.42553**	
2	**3.49838**	**3.53758**	
3	**3.53758**	**3.55904**	
:	**:**	**:**	
12	**3.58533**	**3.58553**	**1.9943 X 10^{-4}**
13	**3.58553**	**3.58564**	**1.11478 X 10^{-4}**
14	**3.58564**	**3.58571**	**0.6231 X 10^{-4}**

- <u>Convergence criteria for fixed-point iteration</u>

➢ Recall in this method we converted $f(x) = 0$ to $x = g(x)$.

➢ Using iteration methods $x_{i+1} = g(x_i)$

➢ Q. Do you think the form $x = g(x)$ will converge to solution every time?

- In the Archimedes's principle equation

$$\rho_f \frac{h^3}{3} - \rho_f R h^2 + \frac{4}{3} R^3 \rho_f = 0$$

If $\rho_f = 1.05$ g/c.c., $\rho_s = 1.25$ g/c.c. and $R = 1.0$ cm, then

$$0.35h^3 - 1.05h^2 + 1.66667 = 0$$

$$\Rightarrow 0.35h^2(h-3) = -1.66667$$

$$\Rightarrow h - 3 = \frac{-1.66667}{0.35h^2}$$

$$\Rightarrow h = 3 - \frac{1.66667}{0.35h^2}$$

$$\Rightarrow h = g(h)$$

- If you start with initial guess say *h = 3.00000,* it may not converge to the solution.
- That is, there is some convergence criteria for this procedure

- Now here $x_{i+1} = g(x_i)$. If α is one of the exact root of $f(x) = 0$ and we are doing iteration to find this α, then Error in $(i+1)^{th}$ iterations $e_{i+1} = x_{i+1} - \alpha$

  i.e. $e_{i+1} = x_{i+1} - \alpha = g(x_i) - g(\alpha)$.

- Let us use Taylor's series for expanding $g(\alpha)$ with respect to the base point *xi*,

i.e. $g(\alpha) = g(x_i) + g'(x_i)(\alpha - x_i) + (1/2!)g''(x_i)(\alpha - x_i)2 + \ldots$

Truncating this series after final differential terms, you can get

$g(\alpha) = g(x_i) + g'(\zeta)(\alpha - x_i)$; where $x_i \le \zeta \le \alpha$

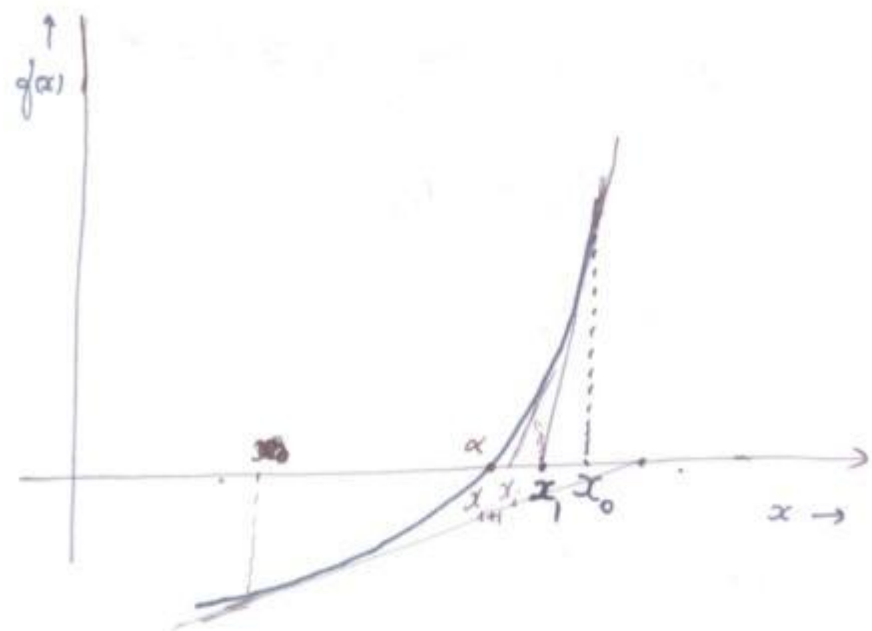
So, $e_{i+1} = g(x_i) - [g(x_i) + g'(\zeta)(\alpha - x_i)]$

$= - g'(\zeta)(\alpha - x_i)$

Note, $e_i = x_i - \alpha$. So, $e_{i+1} = g'(\zeta)\, e_i$

- For this iteration process to converge, we need to have $e_{i+1} < e_i$ or, $| e_{i+1} / e_i | = g'(\zeta) < 1.000$
- If $| e_{i+1} / e_i | > 1.000$, then iteration diverges (no solution will be obtained).
- <u>Newton's Method:</u>
- Also called Newton-Raphson method.
- One of the most well known and widely used open domain method for solving non-linear equations.

- You start with initial guess say x_o.
- Through the graphical point $(x_o, f(x_o))$ a tangent to the curve $f(x)$ is drawn.
- This tangent intersects the x-axis at x_1. This is improved point of x.



- Again a tangent is drawn at $(x_1, f(x_1))$
- Similar procedure continued till $xi+1$ converges with the exact solution α.
- At any (i+1)th iteration, x_{i+1} is obtained.
- At convergence $f(x_{i+1}) \rightarrow f(\alpha) = 0$.
- To obtain the improved point $xi+1$ from the previous point x_i, we can use straight line principle.
- Slope of tangential straight line passing $(x_i, f(x_i))$ is

  $g'(x_i) = (g(x_{i+1} - g(x_i))/(x_{i+1} - x_i) = (0 - f(x_i))/(x_{i+1} - x_i)$.
- Note that the slope of the straight line $g(x)$ passing through $(x_i, f(x_i))$ is same as slope of the curve passing through $(x_i, f(x_i))$.

  So, $g'(x_i) = f'(x_i) = (0 - f(x_i))/(x_{i+1} - x_i)$

  or, $x_{i+1} = x_i - f(x_i)/ f'(x_i)$.
- The process repeated till convergence $|x_{i+1} - x_i| \leq \varepsilon$

  or. $|f(x_{i+1})| \leq \varepsilon$ etc.

- Example: Solve the function $f(x) = x^2 – 10 x + 23$ using Newton-Raphson method.
- Recall while doing regula-falsi method we identified that the root lies in between [0,4]. But it took nearly 10 iterations to arrive at the solution. That was because it was closed domain approach.
- Here let us start with initial guess $x_0 = 0$.

$f(x) = x^2 – 10 x + 23$ and $f'(x) = 2x – 10$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	$\|x_{i+1} - x_i\|$
0	1.00000	14.0000	-8.0000	2.75000	1.7500
1	2.75000	3.06250	-4.5000	3.43055	0.68055
2	3.43055	0.46317	-3.1389	3.57811	0.14756
3	3.57811	0.021771	-2.84378	3.58576	0.00765
4	3.58576	7.48×10^{-5}	-2.82848	3.58578	2×10^{-5}

- You can see that the solution here converged in 4 iterations. That is N-R method is fast. Why?