LECTURE
18-JAN-2014

FUNDAMENTAL TRANSPORT EQUATIONS

Yesterday we started discussion on properties for fluid continuum.

For a heterogeneous multi-species liquid
* Considering a volume dU in that fluid

we had defined

→ species density, ρ_α
→ density of the entire fluid, ρ
→ Mass fraction, $\omega_\alpha = \rho_\alpha / \rho$

Now at each mathematical point in the space, we may be interested to define velocity. Consider a mathematical point P within the elementary volume dU.

→ We can define specie velocity $\vec{V}_\alpha$ (The statistical average velocity of specie α in the volume dU) $= \dfrac{\text{Sum of velocities of individual molecules of } \alpha\text{-sp}}{\text{Total no. of molecules of } \alpha\text{-specie.}}$

→ We can also define:

$$\vec{V}^* = \frac{\sum_{\alpha=1}^{N} \rho_\alpha \vec{V}_\alpha}{\sum_{\alpha=1}^{N} \rho_\alpha} = \frac{\sum_{\alpha=1}^{N} \rho_\alpha \vec{V}_\alpha}{\rho}$$

This is mass average velocity ($\vec{V}^*$).

$\rho_\alpha \vec{V}_\alpha$ → Mass flux (Mass passing through unit area per unit time).

$\vec{v}^*$ interpretation $\rightarrow$ Momentum per unit mass of the flowing fluid.

$\rho \vec{v}^*$ is momentum per unit volume of the multi-specie fluid.

$$\rho \vec{v}^* = \sum_{\alpha=1}^{N} \rho_\alpha \vec{v_\alpha}$$

i.e. Momentum per unit volume of the fluid = Sum of momenta ~~of~~ per unit volume of individual species.

⇒ Volume average velocity

We can also define volume averaged velocity for a multi-specie heterogeneous fluid.

$$\vec{v}' = \sum_{\alpha=1}^{N} \rho_\alpha u_\alpha \vec{v}_\alpha$$

Here $\vec{v}'$ $\rightarrow$ volume averaged velocity in the element dV for the entire fluid.

u_α $\rightarrow$ partial specific volume of α-specie.

$$u_\alpha = \frac{\partial U}{\partial m_\alpha}$$

Note: $$\sum_{\alpha=1}^{N} \left(\frac{\partial U}{\partial m_\alpha}\right) \frac{dm_\alpha}{dU} = 1 = \sum_{\alpha=1}^{N} \rho_\alpha u_\alpha$$

Again note that for a homogeneous fluid

$\alpha = 1$

$\therefore \vec{v_\alpha} = \vec{v}^* = \vec{v}'$

Till now, we defined velocity in a heterogeneous fluid based on the property <u>mass</u>.

→ The fluid molecules can have some other properties associated like

* Momentum, kinetic energy, etc.

→ As we determined for the element fluid volume dU, the mass averaged velocity $\vec{V}^*$, we can also define momentum averaged velocity $\vec{V}_{mom}$, or kinetic ~~ener~~ energy averaged velocity $\vec{V}_{KE}$.

Now $\vec{V}^* \neq \vec{V}_{mo} \neq \vec{V}_{KE}$

→ Again the fluid particle can be defined using various properties. Apart from having fluid particles based on mass (or density), we can also define fluid particles based on momentum, kinetic energy, etc.

* If we follow these properties, the pathlines for those properties can be obtained.

The Extensive Properties

Extensive properties depend on mass of the fluid to which these properties are associated.

e.g. Mass, Volume, Energy, momentum and kinetic energy.

These properties may be
→ Scalar
↳ Vector
↳ Tensor

We symbolically represent the extensive property as G.
∴ A particle of fluid (or continuum) defined based on G can have instantaneous velocity $\vec{V}_G$.
→ In a multi-specie liquid we can also define property G_α, the corresponding velocity is given as $\vec{V}_{G\alpha}$

The Intensive Property

Amount of the property per unit mass of the fluid is called intensive property. Symbol used is γ.
In a specie α, we can also have γ_α

As we have mass density ρ (or ρ_α) we can also have density for the ~~so~~ said fluid property. (Symbol used is g).

i.e. $$g = \frac{\text{Amount of property } G \text{ in fluid}}{\text{Volume of fluid}}$$

or $$g_\alpha = \frac{\text{Amount of property } G \text{ in } \alpha\text{-specie in fluid}}{\text{Volume of fluid}}$$

For any volume U we can then write:

$$G_\alpha (x, t) = \int_{[U(\vec{x})]} \gamma_\alpha \rho_\alpha \, dU = \int_U g_\alpha \, dU .$$

$\Rightarrow$ Velocity $\vec{V}_G$ of an extensive property G in a multi-specie liquid is given by:

$$\vec{V}_G = \frac{\sum_{\alpha=1}^{N} (g_\alpha \vec{V}_{G\alpha})}{\sum_{\alpha=1}^{N} g_\alpha}$$

$\vec{V}_{G\alpha}$ $\rightarrow$ velocity of propagation of property G in α-specie portion of the fluid.

<u>Diffusive Velocities & Fluxes</u>

Each fluid particle (based on the property considered) have its own pathline in the flow domain.

For that the identity of the fluid particle is preserved during a finite period of time.

Most of the conservative properties for these particles are based on molecular diffusion.

If we have $\vec{V}_\alpha \to$ velocity of specie α

$\vec{V}^* \to$ ~~Ave~~ Mass averag velocity

$\vec{V}' \to$ Volume average velocity

Then $\vec{V}_G - \vec{V}' = \hat{\vec{V}}'_\alpha$

$\hat{\vec{V}}'_\alpha \to$ diffusive velocity of α-specie

$\hat{\vec{V}}^*_G = \vec{V}_G - \vec{V}^*$

$\hat{\vec{V}}^*_{G\alpha} = \vec{V}_{G\alpha} - \vec{V}^*$

Then we can define diffusive velocities

$\vec{V}_\alpha - \vec{V}' = \hat{\vec{V}}'_\alpha$ (Diffusive velocity w.r.t volume average velocity)

Also $\vec{V}_\alpha - \vec{V}^* = \hat{\vec{V}}^*_\alpha$ (Diffusive velocity w.r.t mass average velocity)

For other properties, we can again form the diffusive velocities as such:

$\hat{\vec{V}}^*_G = \vec{V}_G - \vec{V}^*$

or $\hat{\vec{V}}^*_{G\alpha} = \vec{V}_{G\alpha} - \vec{V}^*$

or $\hat{\vec{V}}'_G = \vec{V}_G - \vec{V}'$

or $\hat{\vec{V}}'_{G\alpha} = \vec{V}_{G\alpha} - \vec{V}'$, etc.

Molecular diffusion is based on linear relation between diffusive mass fluxes and the driving forces producing the mass movement. Similarly for other properties you can associate relationship between diffusive property fluxes and the driving forces that produce the movement of these properties.

$\therefore$ We can suggest concepts on <u>Diffusive Mass Flux</u>

eg: For α-specie in a heterogeneous fluid

$$\vec{J}_\alpha^* = \rho_\alpha \hat{\vec{V}}_\alpha^* = \rho_\alpha (\vec{V}_\alpha - \vec{V}^*)$$

$$\vec{J}_\alpha' = \rho_\alpha \hat{\vec{V}}_\alpha' = \rho_\alpha (\vec{V}_\alpha - \vec{V}')$$

Also note that

$$\sum_{\alpha=1}^{N} \vec{J}_\alpha^* = 0 \qquad (\text{Why ?})$$

$\Rightarrow$ Extending this concept for other properties, we can have <u>diffusive fluxes</u>.

$$\vec{J}_{G\alpha}^* = g_\alpha \hat{\vec{V}}_{G\alpha}^* = g_\alpha (\vec{V}_{G\alpha} - \vec{V}^*)$$

$$\vec{J}_G^* = g \hat{\vec{V}}_G^* = g (\vec{V}_G - \vec{V}^*)$$

Also we have $\vec{J}_{G\alpha}' = g_\alpha \hat{\vec{V}}_{G\alpha}' = g_\alpha (\vec{V}_{G\alpha} - \vec{V}')$

$$\vec{J}_G' = g \hat{\vec{V}}_G' = g (\vec{V}_G - \vec{V}')$$

The Lagrangian and Eulerian Approach for flow analysis

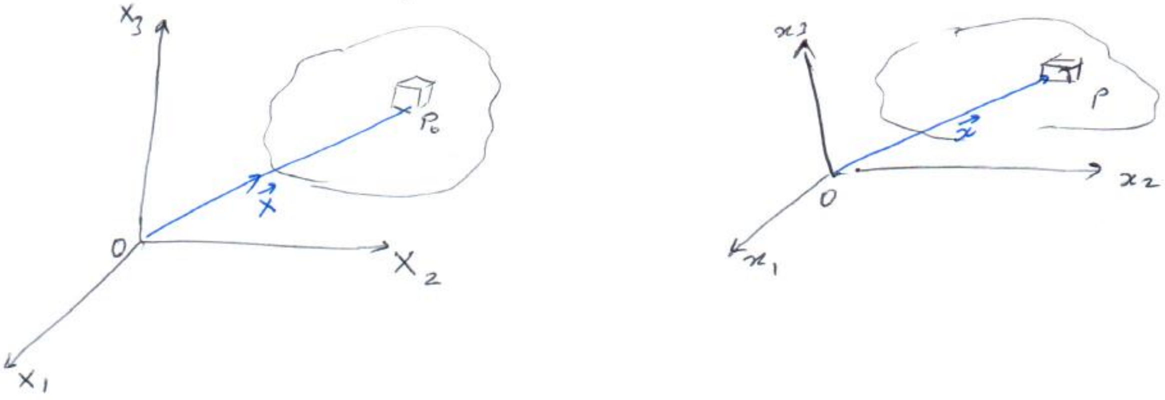
Q: What is Lagrangian method ?

Q: What is Eulerian method ?

Lagrangian method is usually adopted in solid mechanics.

Consider a particle in ~~of~~ a continuum

time, $t = 0$ time, $t = t$



Initially at time $t=0$, the position of particle in the continuum P_0 is given by the position vector. The position is represented in <u>material co-ordinates</u> $O\,X_1 X_2 X_3$.

$$\vec{X} = X_1 \hat{I}_1 + X_2 \hat{I}_2 + X_3 \hat{I}_3$$

After a certain time, the ~~pa~~ particle along with continuum has moved. But still we can represent its position using the new co-ordinate $O\,x_1 x_2 x_3$. ~~Th~~ In solid mechanics it is easier to express like that. The co-ordinate $O\,x_1 x_2 x_3$ called <u>spatial co-ordinates</u>.

$$\vec{x} = x_1 \hat{i}_1 + x_2 \hat{i}_2 + x_3 \hat{i}_3$$

$\therefore$ Now one can also describe

$$\vec{x} = \vec{x}(\vec{X}, t)$$

That is spatial coordinate is function of material coordinate.
Such approach is Lagrangian.

→ If we merge the origins of both co-ordinates,
(in fluid mechanics we need to do that)

i.e. $$\vec{x} = \vec{x}(\vec{X}, t)$$

and $$\vec{x} = \vec{x}(\vec{X}, 0) = \vec{X}$$

★ We should have $\vec{x}$ and $\vec{X}$ continuously differentiable. (The meaning of differentiable is that two particles in the continuum if they were distinct initially will remain distinct for all time).

$\therefore$ If $$\vec{x} = \vec{x}(\vec{X}, t)$$

Then we can invert this relationship to get the material coordinate (or initial position) of the particle.

i.e. $$\vec{X} = \vec{X}(\vec{x}, t)$$

This will be Eulerian approach.

★ Necessary and sufficient condition for the existence of such inverse functions

↳ Jacobian $J \neq 0$

where $$J \equiv \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

or $$J = \left| \frac{\partial x_i}{\partial X_j} \right|$$

$\Rightarrow$ When we described the spatial co-ordinate

$$\vec{x} = \vec{x}(\vec{X}, t)$$

we are actually describing the pathline of a particle whose initial position was $\vec{X}$.

$\Rightarrow$ For a multi-specie system consisting of N-components $\alpha = 1, 2, 3, \ldots, N$ then we can also define the initial position of any mass particle of specie α.

i.e. We can give position vector $\vec{X}_\alpha$

During its motion, the position changes and we can give the instantaneous velocity of the α-specie mass particle as

$$\vec{V}_\alpha(\vec{X}_\alpha, t) = \left. \frac{\partial \vec{x}}{\partial t} \right|_{\vec{X}_\alpha = \text{constant}}$$

$\left(\because \vec{X}_\alpha = \text{constant} \right.$ material position initially $\left. \right)$

$\rightarrow$ i.e. It is the rate of change of position $\vec{x}$ of the mass particle of α-specie whose initial position is at $\vec{X}_\alpha$

In the Eulerian approach, we get instantaneous picture of the velocities at all points in the continuum.

The Eulerian velocity $\vec{V}(\vec{x}, t)$.

i.e.
$$V_{x_1} = V_{x_1}(x_1, x_2, x_3, t)$$
$$V_{x_2} = V_{x_2}(x_1, x_2, x_3, t)$$
$$V_{x_3} = V_{x_3}(x_1, x_2, x_3, t)$$

⇒ Both the approach used to determine the velocities of particles based on other properties. as well.

The Substantial Derivative

Recall in Lagrangian approach $\vec{V}_\alpha(\vec{x}_\alpha, t) = \left.\frac{\partial}{\partial t}\vec{x}\right|_{\vec{x}_\alpha = \text{constant}}$

→ Similar such partial differentiations w.r.t any fluid property following a particle of that property we need to keep $\vec{x}_\alpha$ = constant. i.e $\left.\frac{\partial(\,)}{\partial t}\right|_{\vec{x}_\alpha = \text{constant}}$.

Now we replace that by $\frac{D(\,)}{Dt}$

This is called Material Derivative, Hydrodynamic Derivative or Total Derivative.

Now recall that we can define particles based on various properties (G).

So you can have mass particle, momentum particle, kinetic energy particle, etc.

$\Rightarrow$ Similarly for a multi-specie component we can define fluid particles based on above different properties for each α-specie (i.e. G_α)

$\Rightarrow$ we can consider a property B_α in the fluid particle defined by G_α :

e.g: For a fluid mass particle (G corresponds to mass),

$\vec{V}^* \rightarrow$ Mass average velocity

$\rho \rightarrow$ Density

$\vec{x} \rightarrow$ position vector of the fluid mass particle.

Now we have $\vec{x} = \vec{x}(\vec{X}, t)$

$$\vec{V}^* = \left.\frac{\partial \vec{x}}{\partial t}\right|_{\vec{X} = \text{constant}} = \frac{D\vec{x}}{Dt}$$

$\Rightarrow$ Similarly using Lagrangian formulation:

$\rightarrow$ The temporal rate of change of B_α

$$\vec{x} = \vec{x}(\vec{X}_\alpha, t)$$

$$\left.\frac{\partial x_i(\vec{X}_\alpha, t)}{\partial t}\right|_{\vec{X}_\alpha = \text{constant}} = \frac{D x_i}{Dt} = (V_{G\alpha})_i$$

As $\quad B_{\alpha} = B_{\alpha}(x_1, x_2, x_3, t)\Big|_{\vec{X}_{\alpha} = \text{constant}}$

$$\therefore \quad \frac{DB_{\alpha}}{Dt} = \frac{\partial B_{\alpha}}{\partial x_1}\frac{\partial x_1}{\partial t}\Big|_{\vec{X}_{\alpha} = \text{constant}} + \frac{\partial B_{\alpha}}{\partial x_2}\frac{\partial x_2}{\partial t}\Big|_{\vec{X}_{\alpha} = \text{constant}}$$

$$+ \frac{\partial B_{\alpha}}{\partial x_3}\frac{\partial x_3}{\partial t}\Big|_{\vec{X}_{\alpha} = \text{constant}} + \frac{\partial B_{\alpha}}{\partial t}$$

$$\text{i.e.} \quad \frac{DB_{\alpha}}{Dt} = \frac{\partial B_{\alpha}}{\partial t} + \frac{\partial B_{\alpha}}{\partial x_1} V_{x_1} + \frac{\partial B_{\alpha}}{\partial x_2} V_{x_2} + \frac{\partial B_{\alpha}}{\partial x_3} V_{x_3}$$

$$= \underbrace{\frac{\partial B_{\alpha}}{\partial t}}_{\text{Local Derivative}} + \underbrace{(\vec{V}_{G\alpha} \cdot \vec{\nabla}) B_{\alpha}}_{\text{Convective Derivative}}$$