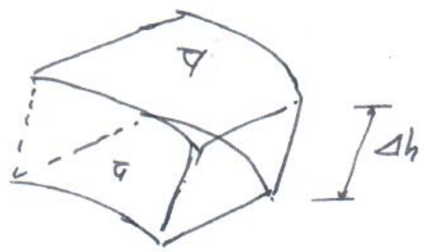
29-MAR-2014

CONTINUITY EQUATION FOR WATER IN UNCONFINED AQUIFERS USING DUPUIT'S APPROXIMATIONS

Yesterday, we derived the basics for continuity equation in unconfined aquifers.

As the phreatic surface varies w.r.t. time, the change in storage of water in the control volume occurs due to raise (or lowering) of saturated zone and compression of aquifer matrix.

We arrived at the equation :



$$\cancel{\rho n_e} \quad \rho n_e \frac{\partial h}{\partial t} + \int_0^h \frac{\partial}{\partial t}(\rho n)\, dz + \frac{\partial}{\partial x}(\rho Q_{wx}) + \frac{\partial}{\partial y}(\rho Q_{wy}) - \rho N = 0 \quad \rightarrow (1)$$

Recall while deriving continuity equation for a vertically consolidating porous medium, we arrived at the equation:

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) + n \frac{\partial \rho}{\partial t} + \rho \frac{\partial n}{\partial t} = 0 \rightarrow (2)$$

(Refer LECTURES 22 and 23)

In Equation (2), it was seen that

$$\frac{\partial}{\partial t}(\rho n) = \rho \left[\beta n + \alpha_b'(1-n) \right] \frac{\partial p}{\partial t}$$

where $\beta \rightarrow$ liquid compressibility

$\alpha_b' \rightarrow$ Coefft. of bulk compressibility for $p =$ constant.

Therefore, equation 2 got modified to

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) = -\rho\left[\beta n + (1-n)\alpha_b'\right]\frac{\partial p}{\partial t}$$

~~Simulated~~ We also defined in LECTURE 23 dated 10-MAR-2014 that specific mass storativity w.r.t. pressure changes can be given as:

$$S_{op}^{*} = \rho\left[\beta n + \alpha_b'(1-n)\right]$$

$$\therefore \frac{\partial}{\partial t}(\rho n) = S_{op}^{*}\frac{\partial p}{\partial t}$$

Similarly for the case of unconfined aquifers

→ In equation 1 we can write:

$$\int_0^h \frac{\partial}{\partial t}(\rho n)\, dz = \int_0^h S_{op}^{*}\frac{\partial p}{\partial t}\, dz$$

Again from Dupuit's assumptions, we have

$$\phi = h + z$$

~~or Datum Lying~~ or hydrostatic condn.

$$p = \rho g h$$

$$\frac{\partial p}{\partial t} = \rho g \frac{\partial h}{\partial t}$$

$$\therefore \int_0^h \frac{\partial}{\partial t}(\rho n)\, dz = \int_0^h S_{op}^{*}\rho g\frac{\partial h}{\partial t}\, dz$$

$$= S_{op}^{*}\rho g h\frac{\partial h}{\partial t}$$

$\therefore$ Equation 1 becomes:

$$\rho n_e\frac{\partial h}{\partial t} + S_{op}^{*}\rho g h\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}(\rho Q_{wx}) - \frac{\partial}{\partial y}(\rho Q_{wy}) + \rho N$$

Or.

$$\rho\left[n_e + S_{op}^{*} g h\right]\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}(\rho q_{wx}) - \frac{\partial}{\partial y}(\rho q_{wy}) + \rho N$$

Substituting $q_{wx} = -K\frac{\partial h}{\partial x}h$ and $q_{wy} = -Kh\frac{\partial h}{\partial y}$

$$\rho\left[n_e + S_{op}^{*} g h\right]\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\rho K h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho K h\frac{\partial h}{\partial y}\right) + \rho N \rightarrow 3$$

In most of the groundwater hydrological unconfined aquifers

$$n_e \ggg S_{op}^{*} g h$$

$\therefore$ Equation 3 becomes

$$\frac{\partial}{\partial x}\left(\rho K h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho K h\frac{\partial h}{\partial y}\right) + \rho N = \rho n_e\frac{\partial h}{\partial t}$$

Again as water is incompressible and porous media being isotropic; we have

$$K\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right] + N = n_e\frac{\partial h}{\partial t}$$

or

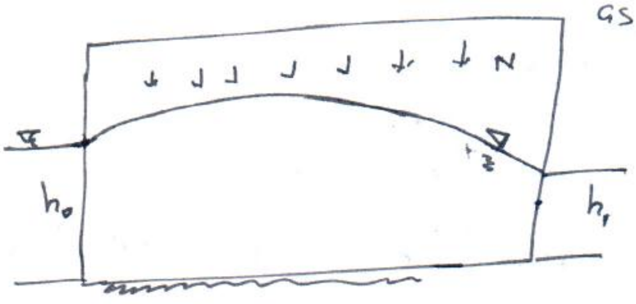
$$\boxed{\frac{K}{2}\left[\frac{\partial^2}{\partial x^2}(h^2) + \frac{\partial^2}{\partial y^2}(h^2)\right] + N = n_e\frac{\partial h}{\partial t}} \rightarrow 4$$

Equation 4 is Forchheimer's equation for unconfined flow using Dupuit's assumptions.

$\Rightarrow$ We can solve Forchheimer's equation for various scenarios.

For example:

You try to solve on your own the Forchheimer equation:



WELL HYDRAULICS

In last few classes we have seen to develop continuity equations for confined aquifers and unconfined aquifers.

→ We have also seen the governing partial differential equations to describe confined and unconfined flows:

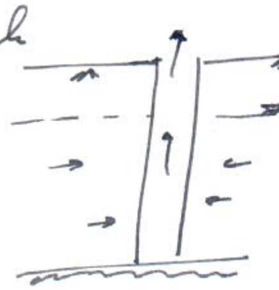
Today we will see about flow of groundwater in wells → can be borewells, or dug wells, or piezometers, etc.

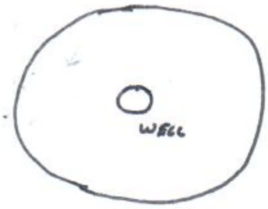
→ We know that human civilisations have utilised the concepts of wells, borewells, tubewells, dug wells, etc. to tap water that exist below the surface of earth.

→ Let us now see about the flow mechanisms in wells. Part of WELL HYDRAULICS.

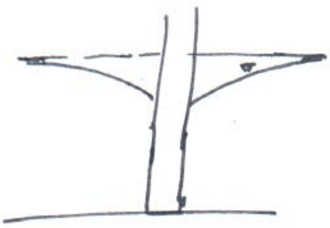
STEADY RADIAL FLOW TO A WELL

→ Water can be pumped from wells for beneficial needs.
→ If water is ~~removed~~ pumped from wells, then adhering to continuity principles, we know that this water is removed from the aquifer.
→ Let us imagine a scenario in which water is pumped ~~continuosly~~ continuously and steady state conditions exist in the location.





* Due to discharge from well a region around the well exist where there will be fluctuations in piezometric surface (or water table) due to pumping. The piezometric surface will lower.

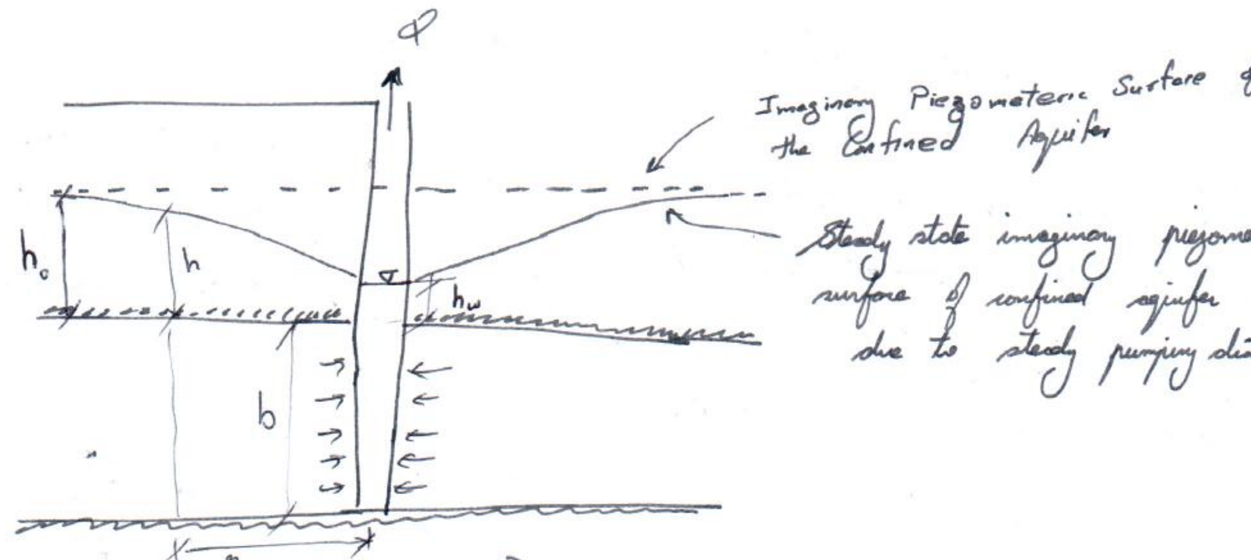


* The lowering of piezometric surface is the drawdown of the surface.

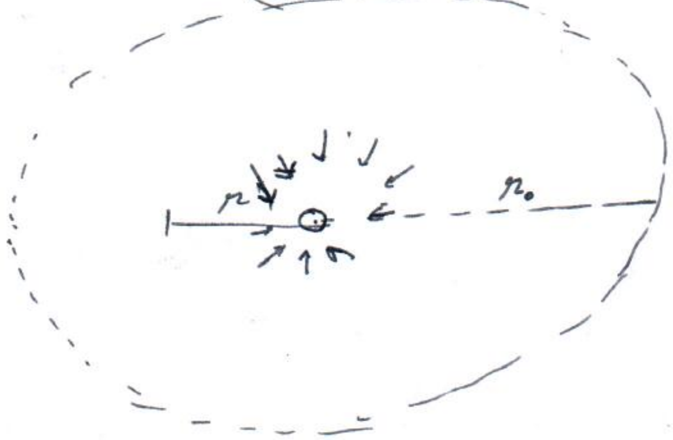
* Through observations one can measure the drawdowns.

Radial Flow In a Confined Aquifer

As discussed earlier we are assuming steady state conditions for radial flow in a well penetrating a confined aquifer.



(Elevation View)



Top View

From the top view, you can see the radius of influence of pumping (i.e. r_0).

Let the diameter of well $= 2 r_w$

Assuming the flow in the confined aquifer is horizontal and the well completely penetrates the confined aquifer,

→ Radial flow occurs into the well due to continuous discharge.

→ In the radial direction:

$$q_r = -K \frac{dh}{dr}$$ (ofcourse for an isotropic porous media).

In steady conditions, $Q = q_r \times \text{Surface Area}$

$$= q_r \cdot 2\pi r b$$

or $$Q = -2\pi r b K \frac{dh}{dr} \rightarrow (1)$$

We can now suggest that $-$ive sign ~~[illegible]~~ shows pumping (or extraction) and $+$ive sign will describe recharge.

We have seen in this steady conditions r_o is the radius of influence.

At $r = r_o$, we have $h = h_o$

At $r = r_w$, we have $h = h_w$

Using this B.C.s, we can solve eqn. (1) as:

$$\frac{Q}{2\pi b K} \int_{r_w}^{r_o} \frac{dr}{r} = -\int_{h_w}^{h_o} dh$$

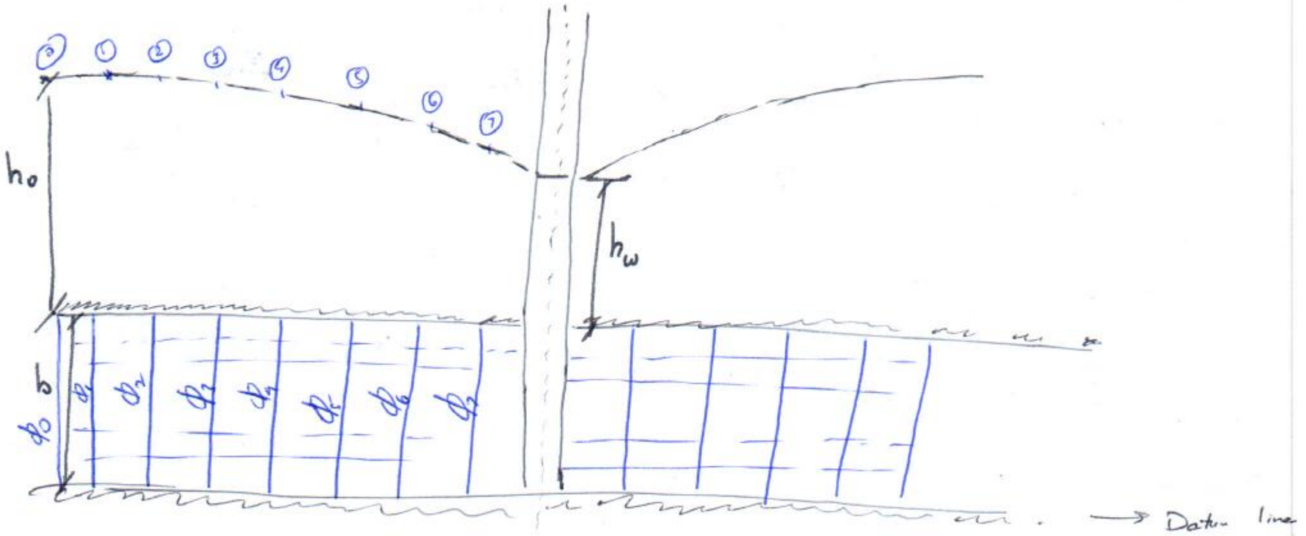
or $$h_o - h_w = -\frac{Q}{2\pi b K} \ln\left(\frac{r_o}{r_w}\right)$$

or Steady discharge. $$Q = \frac{-2\pi b K (h_o - h_w)}{\ln (r_o/r_w)}$$

$\therefore$ Solution for (1) will be

$$Q = -\frac{2\pi b K (h - h_w)}{\ln (r/r_w)}$$

The lowering of piezometric surface indicate that there will be reduction in piezometric head from far distance to the well



If the datum line is given as below:

At portion 0 → piezometric head $\phi_o = h_o + b$

At portion 1 → $\phi_1 = h_1 + b$

2 → $\phi_2 = h_2 + b$ ...

then $\phi_0 > \phi_1 > \phi_2 > \phi_3$ ...

That is piezometric head will be same along a vertical line as given in blue colours.

We can draw equipotential lines as given in blue colours.

⇒ Perpendicular to equipotential lines will be streamlines or flow lines. Together we will get flow net.

⇒ The equation $\boxed{Q = -\dfrac{2\pi b K (h - h_w)}{\ln (r/r_w)} = -\dfrac{2\pi T (h - h_w)}{\ln (r/r_w)}}$

is called Thiem's equation to determine transmissivity of a confined aquifer.