LECTURE 23

10 - MARCH - 2014

CONTINUITY EQUATION OF A HOMOGENEOUS COMPRESSIBLE FLUID IN A DEFORMING POROUS MEDIA

In the last we discussed that in a deforming porous media, the continuity equation of a homogeneous compressible fluid will be:

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) + n\frac{\partial \rho}{\partial t} + \rho\frac{\partial n}{\partial t} = 0 \quad \to (1)$$

i.e. $\frac{\partial n}{\partial t} \neq 0$

To evaluate $\frac{\partial n}{\partial t}$, we discussed on coefficient of bulk compressibilities

Adopting the relations between σ, σ', and p and for a linearly compressing saturated porous media, we found that

$$\frac{1}{(1-n)^2}\frac{\partial n}{\partial t} = -\alpha^*\left(\frac{\partial \sigma}{\partial t} - \frac{\partial p}{\partial t}\right) \quad \to (2)$$

⇒ As the fluid is compressible, we also assumed that the density of liquid $\rho = \rho(p)$.
(i.e. it is function of pressure only)

Recall in iso-thermal conditions, the fluid compressibility

$$\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \quad ; \quad T = \text{constant}$$

$$\therefore \quad \frac{\partial \rho}{\partial t} = \beta \rho \frac{\partial p}{\partial t} \quad \rightarrow (3)$$

Also from (2);

$$\frac{\partial n}{\partial t} = -(1-n)^2 \alpha^* \left(\frac{\partial \sigma}{\partial t} - \frac{\partial p}{\partial t} \right) \quad \rightarrow (4)$$

Substituting (3) and (4) in equation (1) we get:

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) + n\beta\rho \frac{\partial p}{\partial t} + \rho \alpha^* (1-n)^2 \left(\frac{\partial \sigma}{\partial t} - \frac{\partial p}{\partial t} \right) = 0$$

i.e.

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) = -\rho \left(n\beta + \alpha^*(1-n)^2 \right) \frac{\partial p}{\partial t} + \rho \alpha^* (1-n)^2 \frac{\partial \sigma}{\partial t} \quad \rightarrow (5)$$

As assumed earlier, in groundwater hydrology, most of the time the total stress σ in subsurface may remain constant.

$$\therefore \quad \frac{\partial \sigma}{\partial t} \approx 0$$

$\therefore$ Equation (5) modifies to:

$$\boxed{\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) = -\rho \left(n\beta + \alpha^*(1-n)^2 \right) \frac{\partial p}{\partial t}} \quad \rightarrow (6)$$

→ The same equation can also be developed using the coefficient of compressibility.

i.e. Define $$\alpha_b' = -\frac{1}{V_b}\frac{\partial V_b}{\partial \sigma'}\bigg|_{p = \text{constant}}$$

$$\therefore \quad \alpha_b' \frac{\partial \sigma'}{\partial t} = -\frac{1}{V_b}\frac{\partial V_b}{\partial t}$$

As $V_b = V_s + V_w$ (recall saturated porous media)

We suggested that individual grains comprising solids are non-compressible.

$\therefore V_s$ = a constant (in a control volume)

$$V_s = (1-n)V_b$$

$$\frac{\partial V_s}{\partial t} = 0$$

i.e. $$\frac{\partial}{\partial t}\left((1-n)V_b\right) = 0$$

or $$\frac{\partial V_b}{\partial t} = \frac{V_b}{(1-n)}\frac{\partial n}{\partial t}$$

$$\therefore \quad \alpha_b' \frac{\partial \sigma'}{\partial t} = -\frac{1}{V_b}\frac{\partial V_b}{\partial t} = -\frac{1}{V_b}\left[\frac{V_b}{(1-n)}\frac{\partial n}{\partial t}\right]$$

i.e. $$\alpha_b' \frac{\partial \sigma'}{\partial t} = -\frac{1}{(1-n)}\frac{\partial n}{\partial t}$$

$$\therefore \quad \frac{\partial n}{\partial t} = -(1-n)\,\alpha_b' \frac{\partial \sigma'}{\partial t}$$

$$= -(1-n)\,\alpha_b'\left(\frac{\partial \sigma}{\partial t} - \frac{\partial p}{\partial t}\right)$$

If $\frac{\partial \sigma}{\partial t} \approx 0.0$, then $$\frac{\partial n}{\partial t} \approx (1-n)\,\alpha_b' \frac{\partial p}{\partial t}$$

→ (7)

∴ Equation (3) and Equation (4) are substituted in equation (1) will give:

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) = -\rho\left[\beta n + \alpha_b'(1-n)\right]\frac{\partial p}{\partial t} \quad \rightarrow (8)$$

⇒ Now let us define a term:

"Specific mass storativity"

This term is related to pressure changes.

$$S_{o_p}^{*} = \rho\left[\beta n + \alpha_b'(1-n)\right] \quad \rightarrow (9)$$

This specific mass storativity related to pressure changes is the change in mass of fluid (or water) in a unit volume of ~~water~~ porous media per unit change in pressure.

⇒ We can also define specific mass storativity using changes in piezometric head.

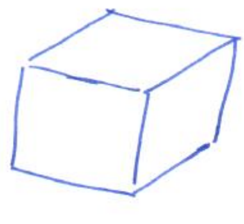
$$S_{o_{\phi^*}}^{*} = \rho^2 g\left[\beta n + \alpha_b'(1-n)\right] \quad \rightarrow (10)$$

Continuity equation for homogeneous compressible fluid in a porous medium, where the solid matrix is in motion

Till now we were suggesting that in the control volume of deforming porous media, the continuity equation is

$$\nabla \cdot (\rho \vec{q}) + \frac{\partial}{\partial t}(n\rho) = 0$$

↳ ⑧



We had taken the specific discharge $\vec{q}$ w.r.t. fixed co-ordinate system.

→ If the solid matrix inside the CV moves, then we need to consider specific discharge w.r.t. fixed co-ordinate system as well as w.r.t. moving solids.

→ Therefore, let us consider the solid grains and the volume associated with solid grains change their position in control volume with a velocity $\vec{V_s}$

$\vec{V_s}$ → velocity of solid volume in the control volume of porous media.

∴ Actual velocity of liquid in CV will be (w.r.t the fixed coordinate system).

$$\vec{V} = \vec{V_r} + \vec{V_s}$$

where $\vec{V_r}$ = Relative velocity of liquid w.r.t the moving solid particles

∴ ~~Equation (8) becomes:~~ $n\vec{V} = n\vec{V}_r + n\vec{V}_s$

The specific discharge. $\vec{q} = \vec{q}_r + n\vec{V}_s$

where $n \rightarrow$ Instantaneous porosity

∴ Equation (8) becomes:

$$\nabla \cdot \left(\rho \left(\vec{q}_r + n\vec{V}_s \right) \right) + \rho \frac{\partial n}{\partial t} + n \frac{\partial \rho}{\partial t} = 0 \quad \rightarrow (9)$$

⇒ We can also the principles in mechanics to develop conservation of mass equation for the solids inside the control volume of the porous media.

Recall for liquid $\nabla \cdot (\rho \vec{q}) + \frac{\partial}{\partial t}(n\rho) = 0$

Similarly for the solids which is in motion:

$$\nabla \cdot \left(\rho_s (1-n) \vec{V}_s \right) + \frac{\partial}{\partial t}\left(\rho_s (1-n) \right) = 0 \quad \rightarrow (10)$$

(x). → As the individual solid grains are non-compressible $\frac{\partial \rho_s}{\partial t} = 0$

$$\therefore \rho_s \nabla \cdot \left((1-n)\vec{V}_s \right) + \rho_s \frac{\partial}{\partial t}(1-n) = 0$$

or $$\nabla \cdot \left((1-n)\vec{V}_s \right) + \frac{\partial}{\partial t}(1-n) = 0$$

or $$\frac{\partial n}{\partial t} = \nabla \cdot \left[(1-n)\vec{V}_s \right]$$

i.e. $$\frac{\partial n}{\partial t} = (1-n)\, \nabla \cdot \vec{V}_s - \vec{V}_s \cdot \nabla n \quad \rightarrow (11)$$

As the continuity equation for liquids is:

$$\nabla \cdot (\rho \vec{q}) + \rho \frac{\partial n}{\partial t} + n \frac{\partial \rho}{\partial t} = 0$$

i.e. $$\nabla \cdot \left[\rho (\vec{q_r} + n \vec{V_s}) \right] + \rho \frac{\partial n}{\partial t} + n \frac{\partial \rho}{\partial t} = 0$$

i.e. $$\nabla \cdot [\rho \vec{q_r}] + \nabla \cdot [\rho_n \vec{V_s}] + \rho (1-n) \nabla \cdot \vec{V_s}$$

$$- \rho \vec{V_s} \cdot \nabla n + n \frac{\partial \rho}{\partial t} = 0$$

Expanding the terms:

$$\nabla \cdot (\rho \vec{q_r}) + \rho n (\nabla \cdot \vec{V_s}) + n \vec{V_s} \cdot \nabla \rho + \rho \vec{V_s} \cdot \nabla n$$

$$+ \rho \nabla \cdot \vec{V_s} - \rho n \nabla \cdot \vec{V_s} - \rho \vec{V_s} \cdot \nabla n + n \frac{\partial \rho}{\partial t} = 0$$

i.e. $$\boxed{\nabla \cdot (\rho \vec{q_r}) + \rho (\nabla \cdot \vec{V_s}) + n (\vec{V_s} \cdot \nabla \rho) + n \frac{\partial \rho}{\partial t} = 0}$$

→ (12)