LECTURE 20

NEW CHAPTER

21 - FEB - 2014

CONTINUITY, CONSERVATION EQUATIONS FOR HOMOGENEOUS FLUIDS IN POROUS MEDIA

In the last class we discussed on equivalent hydraulic conductivities for homogeneous fluids in layered porous media.

→ The meaning of homogeneous fluid – single species in the fluid. ∴ Density of the fluid depends only on pressure (for compressible fluids).

→ In the chapter FUNDAMENTAL TRANSPORT EQUATIONS we had derived conservation equations for multi-specie fluids in porous media. The averaging of properties were done over REV of the porous media and subsequently formulated the transport equations.

→ Here we will see for a homogeneous liquid, how the conservation equation looks?
This topic is more useful for our groundwater hydrology portion as the groundwater is mostly considered homogeneous in a region.

→ In the previous chapter, we also discussed on equation of motion of homogeneous fluid in porous media.

Recall $q_i = -K_{ij} \frac{\partial \phi}{\partial x_j}$

$$q_i = -\frac{k_{ij}}{\mu}\left(\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}(\rho g z)\right)$$

$$q_i = -\frac{k_{ij}\rho g}{\mu} \frac{\partial}{\partial x_i}\left(\frac{p}{\rho g} + z\right)$$

We will use the control volume approach now to develop conservation equations for homogeneous fluids.

⇒ Recall the difference between Lagrangian and Eulerian approaches.

Control Volume :→ A definite volume in space that is fixed.

→ CV is appropriate for flow analysis in porous media (∵ most of the solid matters are not moving in porous media)

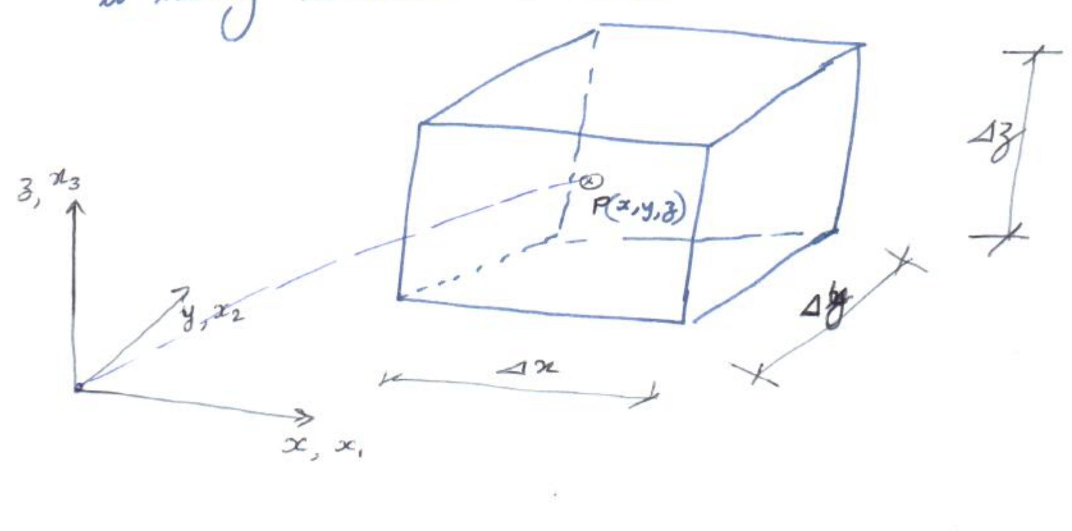
→ CV shape can be of any form → circle, square, rectangle, cube, ellipsoid, etc.

→ The amount of fluid in the CV may change over a time. However the shape and position of CV is assumed same.

Mass Conservation in a Non-deformable Porous Media

Consider a rectangular prism shaped control volume as shown in figure below. The CV that is considered need not be the REV of the porous media. That is the CV may contain several thousands or millions of REVs inside. We are not bothered about that. [Note: If a porous media domain is discretised into finite-difference grid cells for using FDM then each grid cell is a CV].

⇒ Let this CV be associated with a mathematical point $P(x, y, z)$ or $P(x_1, x_2, x_3)$ in space. That is the CV is having P as its centre and is having coordinate (x, y, z).



→ The volume of the above control shape $= \Delta x \, \Delta y \, \Delta z$

⇒ The co-ordinates are chosen in such a direction that the anisotropic properties of the porous media (i.e permeability) or porous media and fluid (i.e. hydraulic conductivity) are ~~taken~~ in principal directions.

So your permeability tensor $k_{ij} = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$

or your hydraulic conductivity tensor

$$K_{ij} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{pmatrix}$$

$\Rightarrow$ We can also think of mass flux through each surfaces of the control volume.

Say in x - direction $\rightarrow$ J_x

y - direction $\rightarrow$ J_y

z - direction $\rightarrow$ J_z

Recall the Reynolds Transport Theorem in general form

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\iiint_{CV} \beta \rho \, dU + \iint_{CS} \beta \rho (\vec{V}\cdot\hat{n})\, dA \quad \rightarrow (1)$$

(Refer LECTURE - ~~7~~ 8 dated 20-JAN-2014, 23-JAN-2014

There we wrote $\oint \frac{DG_\alpha}{Dt} = \int_U \frac{\partial g_\alpha}{\partial t} dU + \int_S g_\alpha (\vec{V}_{G\alpha} \cdot \hat{n})\, dS$)

$\therefore$ As the volume considered here i.e. $\Delta x \Delta y \Delta z$ is not changing w.r.t. time, we can write

$$\int_U \frac{\partial}{\partial t}(\;)\, dU = \frac{\partial}{\partial t}\int_U (\;)\, dU .$$

Let B = mass of fluid in CV = m

ρ = density of the homogeneous fluid

β = intensive property = 1 (for B = mass)

∴ Equation 1 becomes:

$$\frac{Dm}{Dt} = 0 = \frac{\partial}{\partial t}\iiint_{cv} \rho \, dU_v + \iint_{cs} \rho(\vec{V}\cdot\hat{n})\, dA_v$$

Now $\rho(\vec{V}\cdot\hat{n}) \rightarrow$ describes the mass flux through control surfaces.

As the CV is porous media of porosity = n

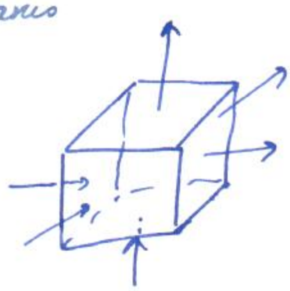
Let the specific discharge be $\vec{q}$

$$\vec{q} = q_1 \hat{i}_1 + q_2 \hat{i}_2 + q_3 \hat{i}_3$$

or

$$\vec{q} = q_x \hat{i} + q_y \hat{j} + q_z \hat{k}$$

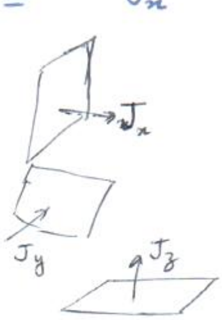
∴ Mass flux through a plane $\perp^r$ to x-axis is as said earlier = J_x



$$J_x = \rho q_x$$

$$J_y = \rho q_y$$

$$J_z = \rho q_z$$



$$\therefore \iint_{cs} \rho(\vec{V}\cdot\hat{n})\, dA = -\rho q_{x-\frac{\Delta x}{2},\,y,z}\,\Delta y \Delta z + \rho q_{x+\frac{\Delta x}{2},\,y,z}\,\Delta y\Delta z$$

$$-\rho q_{y-\Delta y/2,\,x,z}\,\Delta x\Delta z + \rho q_{y+\Delta y/2,\,x,z}\,\Delta x \Delta z$$

$$-\rho q_{z-\Delta z/2,\,x,y}\,\Delta x\Delta y + \rho q_{z+\Delta z/2,\,y,x}\,\Delta x\Delta y$$

i.e. $$\frac{\partial}{\partial t}\iiint_{cv} \rho(n\,dU) + \iint_{cs} \rho(\vec{V}\cdot\hat{n})\,dA_v = 0$$

i.e. $$\frac{\partial}{\partial t}(n\rho)\,\Delta x\Delta y\Delta z + \rho\left[q_{x+\frac{\Delta x}{2},y,z} - q_{x-\frac{\Delta x}{2},y,z}\right]\Delta y\Delta z$$

$$+\rho\left[q_{y+\frac{\Delta y}{2},x,z} - q_{y-\frac{\Delta y}{2},x,z}\right]\Delta z\Delta x + \rho\left[q_{z+\frac{\Delta z}{2},y,x} - q_{z-\frac{\Delta z}{2},x,y}\right]\Delta x\Delta y$$

$$= 0 \qquad \hookrightarrow (2)$$

i.e.

In x- direction in the control volume,

* Net outflux of mass in x- direction

$$= J_x\Big|_{x+\frac{\Delta x}{2},y,z} - J_x\Big|_{x-\frac{\Delta x}{2},y,z}$$

(Mass flux is mass passing through unit area per unit time)

|||ly Net outfluxes in y and z- directions are:

$$J_y\Big|_{x,y+\frac{\Delta y}{2},z} - J_y\Big|_{x,y-\frac{\Delta y}{2},z}$$

and $$J_z\Big|_{x,y,z+\frac{\Delta z}{2}} - J_z\Big|_{x,y,z-\frac{\Delta z}{2}}$$

Using Taylor's series keeping the base point at (x,y,z)

$$J_x\Big|_{x+\frac{\Delta x}{2},y,z} = J_x + \frac{\Delta x}{2}\frac{\partial J}{\partial x} + \left(\frac{\Delta x}{2}\right)^2\frac{1}{2!}\frac{\partial^2 J}{\partial x^2} + \cdots \qquad \hookrightarrow (3)$$

Ily $$J_x \Big|_{x-\frac{\Delta x}{2}, y, z} = J_x - \frac{\Delta x}{2}\frac{\partial J_x}{\partial x} + \frac{\Delta x^2}{8}\frac{\partial^2 J_x}{\partial x^2} - \dots \rightarrow (4)$$

Equation (3) − Eqn. (4) gives

$$J_x \Big|_{x+\frac{\Delta x}{2}, y, z} - J_x \Big|_{x-\frac{\Delta x}{2}, y, z} = \Delta x \frac{\partial J_x}{\partial x} \rightarrow (5)$$

i.e. $$\rho q_x \Big|_{x+\frac{\Delta x}{2}, y, z} - \rho q_x \Big|_{x-\frac{\Delta x}{2}, y, z} = \Delta x \frac{\partial}{\partial x}(\rho q_x) \rightarrow (6)$$

Ily $$\rho q_y \Big|_{x, y+\frac{\Delta y}{2}, z} - \rho q_y \Big|_{x, y-\frac{\Delta y}{2}, z} = \Delta y \frac{\partial}{\partial y}(\rho q_y)$$

and $$\rho q_z \Big|_{x, y, z+\frac{\Delta z}{2}} - \rho q_z \Big|_{x, y, z-\frac{\Delta z}{2}} = \Delta z \frac{\partial}{\partial y}(\rho q_z)$$

Substituting these in eqn. (2), we get.

$$\frac{\partial}{\partial t}(n\rho)\,\Delta x\,\Delta y\,\Delta z + \frac{\partial}{\partial x}(\rho q_x)\Delta x\,\Delta y\,\Delta z + \frac{\partial}{\partial y}(\rho q_y)\,\Delta x\,\Delta y\,\Delta z$$
$$+ \frac{\partial}{\partial z}(\rho q_z)\,\Delta x\,\Delta y\,\Delta z = 0$$

Removing $\Delta x\,\Delta y\,\Delta z$ in the expression we get:

$$\frac{\partial}{\partial x}(\rho q_x) + \frac{\partial}{\partial y}(\rho q_y) + \frac{\partial}{\partial z}(\rho q_z) + \frac{\partial}{\partial t}(n\rho) = 0$$

In Index Notation:

$$\boxed{\frac{\partial}{\partial x_i}(\rho q_i) + \frac{\partial}{\partial t}(n\rho) = 0} \rightarrow (7)$$

In vector notation :

$$\boxed{\frac{\partial}{\partial t}(n \rho) + \nabla \cdot (\rho \vec{q}) = 0}$$

⇒ This is the continuity or conservation of mass equation applicable to a CV of porous media for a homogeneous fluid.

Note: If fluid is incompressible :

Then $\nabla \rho = 0$

$$\frac{\partial \rho}{\partial t} = 0$$

Then the equation of continuity becomes:

$$\boxed{\nabla \cdot \vec{q} = 0}$$

or

$$\boxed{\frac{\partial q_i}{\partial x_i} = 0}$$

Note: In non-deformable porous media, $\frac{\partial n}{\partial t} = 0$

∴ Equation (7) becomes : $n \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho q_i) = 0$

i.e. $n \frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0$

In many practical cases: $\vec{q} \cdot \nabla \rho <<< n \frac{\partial \rho}{\partial t}$ and neglected

$$\therefore \quad \underline{n \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{q} = 0}$$