LECTURE 39
18 - APRIL - 2013

GROUNDWATER HYDROLOGY . . . .

Yesterday, we have seen some basic concepts on groundwater hydrology.

→ Aquifers
→ Specific retention
→ Specific Yield, etc.
→ Storage coefft.

Today we will see how to develop groundwater flow equation.

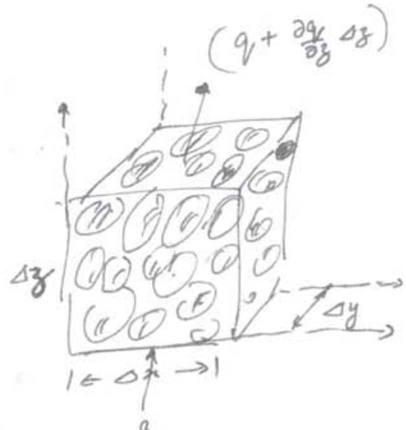
Recall the conservation of mass equation in a porous media using Reynold's Transport theorem

$$\frac{DB}{Dt} = \frac{d}{dt}\iiint_{cv} \beta \rho \, dU + \iint_{cs} \beta \rho \, \vec{v}\cdot\hat{n} \, dA$$

$B \Rightarrow$ mass of water in liquid state.

$\rho \Rightarrow$ density of water

$\beta = 1 =$ intensive property.



$$0 = \frac{d}{dt}\iiint_{cv} \rho \, dU + \iint_{cs} \rho \, \vec{v}\cdot\hat{n} \, dA$$

i.e. $$\iint_{cs} \rho \, \vec{v}\cdot\hat{n} \, dA = -\frac{d}{dt}\iiint_{cv} \rho \, dU$$

$$\iint_{cs} \rho \, \vec{v}\cdot\hat{n} \, dA = \rho\left(q + \frac{\partial q}{\partial z}\Delta z\right)\Delta x \Delta y - \rho q \Delta x \Delta y$$

$$= \rho \frac{\partial q}{\partial z} \Delta x \Delta y \Delta z$$

If we have flow in all three direction and Darcy velocities q_x, q_y, q_z being components

Then

$$\iint_{cs} \rho\, \vec{V}\cdot\hat{n}\, dA = \rho\left[\frac{\partial q_x}{\partial x}\Delta x\Delta y\Delta z + \frac{\partial q_y}{\partial y}\Delta x\Delta y\Delta z + \frac{\partial q_z}{\partial z}\Delta x\Delta y\Delta z\right]$$

$$= \rho\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right]\Delta x\,\Delta y\,\Delta z$$

$\frac{d}{dt}\iiint_{cv} \rho\, dV \rightarrow$ Time rate of change of mass of water stored in the control volume.

→ This change can be due to

* Expansion or compression of water in CV
* Change in pore volume in CV due to expansion or compression of porous CV.

For confined aquifers, it is observed that

~~Let us define a term~~

$$S = \rho g \varepsilon \beta b + \rho g b \eta \quad \text{(Storage Coefft).}$$

where
- $\rho \rightarrow$ density of water
- $g \rightarrow$ acceleration due to gravity
- $\varepsilon \rightarrow$ porosity
- $\beta \rightarrow$ compressibility of water
- $\eta \rightarrow$ compressibility of porous medium
- $b \rightarrow$ aquifer thickness

∴ We can define a term called

Specific Storage (S_s) such that it is the volume of water released from the storage from a unit volume of aquifer due to a unit decrease in piezometric head. (Usually $S_s \approx 1 \times 10^{-4}\ m^{-1}$).

$$\therefore \quad S = S_s\, b$$

$\therefore$ Now we can see that change in mass stored in CV may be due to effects in

→ compression (or expansion) of water

→ compression (or expansion) of pores in CV

$$\frac{d}{dt}\iiint_{CV} \rho\, dV = \rho\, S_s \frac{\partial H}{\partial t}\, \Delta x \Delta y \Delta z$$

$\therefore$ Our mass equation becomes:

$$\rho\, S_s \frac{\partial H}{\partial t}\, \Delta x \Delta y \Delta z + \rho \left[\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial q}{\partial z}\right] \Delta x \Delta y \Delta z = 0$$

or
$$S_s \frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial q}{\partial z} = 0$$

Apply Darcy's law

$$q_x = -K_x \frac{\partial H}{\partial x}$$

$$q_y = -K_y \frac{\partial H}{\partial y}$$

$$q_z = -K_z \frac{\partial H}{\partial z}$$

$$\therefore \boxed{\frac{\partial}{\partial x}\left(K_x \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial H}{\partial z}\right) = S_s \frac{\partial H}{\partial t}}$$

This is the governing equation for flow of ground water.

If you have some source or sink term $W \rightarrow$ discharge of water into or out of CV per unit volume, then

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial H}{\partial z}\right) = S_s \frac{\partial H}{\partial t} + W$$

For isotropic porous medium $K_x = K_y = K_z = K$.

Then $$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = \frac{S_s}{K}\frac{\partial H}{\partial t} + \frac{W}{K}$$

For horizontal confined aquifer $S = S_s b$, $T = K b$

Then $$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{S}{T}\frac{\partial H}{\partial t}$$

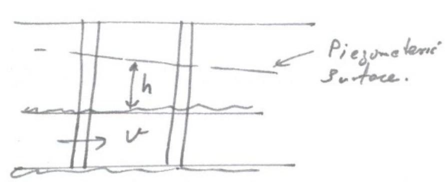
You can infer what happens in steady state.

Cases

1) One-dimensional horizontal flow (steady state).

$$\frac{\partial^2 H}{\partial x^2} = 0$$

or $H = C_1 x + C_2$

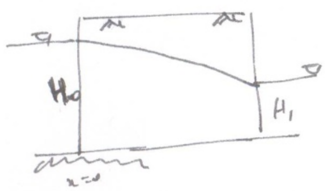


$H = 0$, when $x = 0$

$\therefore \frac{\partial H}{\partial x} = -v/K$

$$\therefore H = \frac{-vx}{K}$$

In unconfined aquifers, you will not get such Laplace equation.



Discharge per unit width $\frac{Q}{B}$

$$\frac{Q}{B} = q_w = -KH\frac{dH}{dx}$$

$$\therefore \quad q_w x = -\frac{K}{2}H^2 + C$$

$H = H_0$ at $x = 0$

$$\therefore \quad q_w = \frac{K}{2x}\left(H_0^2 - H^2\right)$$