LECTURE 31
28-MARCH-2013

Kinematic Wave Routing

In the last class, we discussed on the analytical method of obtaining solutions to kinematic wave problem.

i.e.

We solved the governing equations

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

$$S_0 = S_f$$

→ We found the kinematic wave celerity.

$$c_k = \frac{dx}{dt} = \frac{dQ}{dA}$$

Today, we will briefly see the finite-difference numerical method approach to solve the kinematic wave equation.

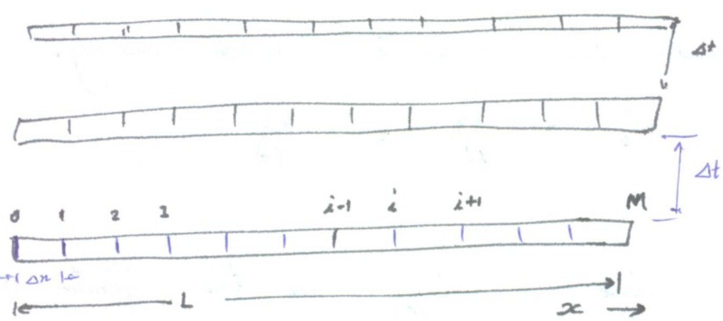
We will briefly say the philosophy of ~~kinematic wave equati~~ finite difference method.

Finite-Difference Method to Solve Kinematic Wave Problem

* FDM is an approximate method to solve pde or ODE.
* The partial derivative or derivatives are approximated by algebraic expression.

* Then these approximations are applied in pde (or ODE) at any location in the domain to convert pde into algebraic equation.
* This algebraic equation is subsequently solved (by matrix methods, etc.)

To illustrate:
Consider a variable T (temperature) that varies w.r.t space and time. For simplicity let us take a one dimensional steel rod



Steel rod

For some analysis we want to find the heat distribution in the steel rod.

As $T(x,t)$, → the heat problem may be functions of T, $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial t}$, $\frac{\partial^2 T}{\partial x^2}$, etc.

→ Now the governing equation will involve the partial derivatives of T and T itself.

So to use finite difference method:

(i) First we will discretise the spatial domain into finite small domain of length Δx

(ii) We will assign node numbers at the interface $i = 1, 2, 3, \ldots, M$

(iii) Similarly we will try to evaluate the temperature at various time steps. That is the entire desired duration is broken into small time steps of Δt duration

(iv) Now the temperature T at any spatial node i and time step s ~~will~~ can be represented as:

$$T_i^{(s)}$$

(v) The spatial partial derivative

$$\left.\frac{\partial T}{\partial x}\right|_i^{(s)} \approx \frac{T_{i+1}^{(s)} - T_{i-1}^{(s)}}{2\Delta x} \quad \text{Centered Difference}$$

$$\text{"} \approx \frac{T_{i+1}^{(s)} - T_i^{(s)}}{\Delta x} \quad \text{Forward difference}$$

$$\text{"} \approx \frac{T_i^{(s)} - T_{i-1}^{(s)}}{\Delta x} \quad \text{Backward difference}$$

(vi) Similarly the temporal derivative

$$\left.\frac{\partial T}{\partial t}\right|_i^{(s)} \approx \frac{T_i^{(s+1)} - T_i^{(s)}}{\Delta t} \quad \text{Forward Time}$$

$$\approx \frac{T_i^{(s)} - T_i^{(s-1)}}{\Delta t} \quad \text{Backward Time.}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_i^{(A)} = \frac{T_{i+1}^{(A)} - 2T_i^{(A)} + T_{i-1}^{(A)}}{\Delta x^2} \quad \text{(Centered Difference).}$$

So in the governing pde, if we approximate the time derivative by forward time approximation → we generally get explicit expression.

* If backward time difference is used, then we get implicit algebraic approximation for the pde..

For the Kinematic Wave

The continuity equation is

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

$$S_0 = S_f$$

i.e. We can write:

$$\frac{\partial Q}{\partial x} + \alpha\beta\, Q^{\beta-1} \frac{\partial Q}{\partial t} = q \longrightarrow (1)$$

Now this is the governing ~~pd equation~~ pde for kinematic wave.

On solving this we will get $Q(x,t)$. for different values of x and time t.

We are now approximating the partial derivatives by linear scheme.

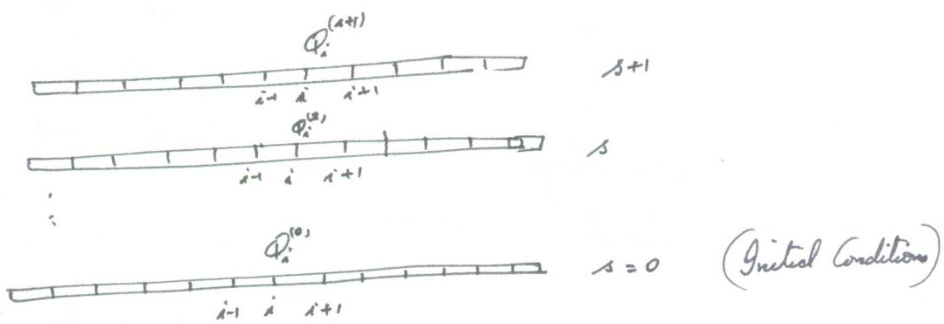
$$\left.\frac{\partial \varphi}{\partial x}\right|_{i}^{(s)} \approx \frac{\Phi_{i+1}^{(s)} - \Phi_{i-1}^{(s)}}{2\Delta x}$$

or We will use forward difference:

$$\left.\frac{\partial \varphi}{\partial x}\right|_{i}^{(s)} = \frac{\Phi_{i+1}^{(s)} - \Phi_{i}^{(s)}}{\Delta x}$$

$$\left.\frac{\partial \varphi}{\partial t}\right|_{i}^{(s)} = \frac{\Phi_{i}^{(s+1)} - \Phi_{i}^{(s)}}{\Delta t}$$

→ Substitute these approximations in pde.



$$\cancel{\Phi_{i+1}^{s}} \left.\frac{\partial \varphi}{\partial x}\right|_{i}^{(s)} + \alpha\beta\left(\Phi_{i}^{(s)}\right)^{\beta-1} \left.\frac{\partial \varphi}{\partial t}\right|_{i}^{(s)} = q_i^{(s)}$$

$$\frac{\Phi_{i+1}^{(s)} - \Phi_{i}^{(s)}}{\Delta x} + \alpha\beta\left(\Phi_{i}^{(s)}\right)^{\beta-1} \frac{\Phi_{i}^{(s+1)} - \Phi_{i}^{(s)}}{\Delta t} = q_i^{(s)}$$

or. $Q_i^{(s+1)}$ = Unknown quantity

$$Q_i^{(s+1)} = Q_i^{(s)} + \frac{q_i^{(s)}}{\alpha\beta (Q_i^{(s)})^{\beta-1}} \Delta t - \frac{\Delta t}{\Delta x} \frac{1}{\alpha\beta (Q_i^{(s)})^{\beta-1}} \left[Q_{i+1}^{(s)} - Q_i^{(s)} \right]$$

Similarly if you take

$$\left.\frac{\partial Q}{\partial x}\right|_i^{(s+1/2)} + \alpha\beta \left(Q_i^{(s+1/2)}\right)^{(\beta-1)} \left.\frac{\partial Q}{\partial t}\right|_i^{(s+1/2)} = q_i^{(s+1/2)}$$

The centered difference comes into picture.

$\cancel{Q_i^{(s+1/2)} =}$

Muskingum - Cunge Method

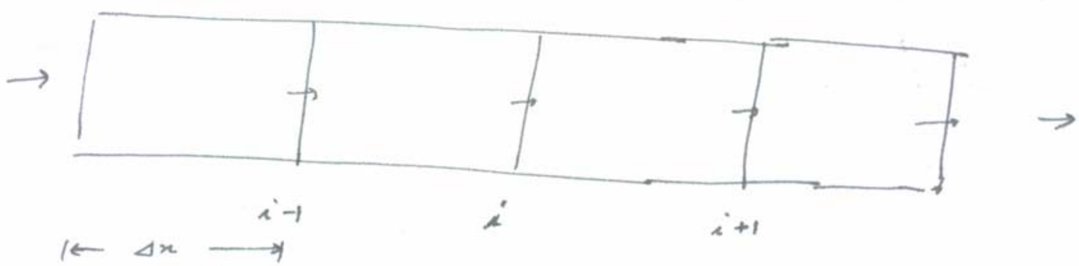
You have seen Muskingum method for lumped flow routing. With same approach and improvement Muskingum - Cunge method can be used to route kinematic wave.

Recall

$$Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j$$

Here the suffix was to express time interval.

In Muskingum - Cunge method what we will do is both spatial and temporal variations are incorporated.



→ The inflow hydrograph is given at $i = 0$.

$$\therefore \quad Q_{i+1}^{(s+1)} = C_1 Q_i^{(s+1)} + C_2 Q_i^{(s)} + C_3 Q_{i+1}^{(s)}$$

C_1, C_2, C_3 → coefficients.

K → storage constant.

DYNAMIC WAVE ROUTING

As you have seen many civilisations developed along river banks.

There are more urbanisations along river banks.

To ~~integrat~~ implement routing we require now more accurate predictions by routing.

$\therefore$ Dynamic wave routing is required.

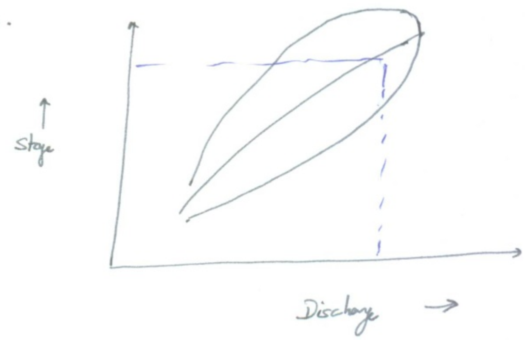
→ If backwater effects are negligible we were able to use kinematic wave approximations.

The dynamic wave equation :

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\left(\frac{\partial y}{\partial x} - S_0 + S_f + S_e\right) - \beta q v_x + W_f B = 0$$

(If all quantities are essential and then incorporated).

When the flow is uniform or $S_0 = S_f$ all other quantities negligible, you have.



i.e. Unique values of Q w.r.t stage.

→ However for dynamic condition, not unique.