LECTURE 27
19-MARCH-2013

DISTRIBUTED FLOW ROUTING (contd...)

In the last class we started discussing on distributed flow routing.

→ The routing is done from u/s to d/s at distributed locations.

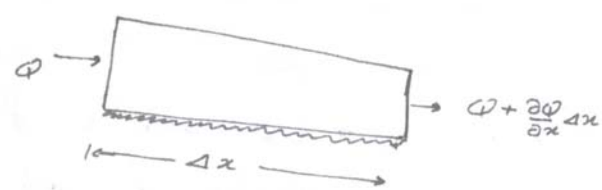
We developed Saint Venant's continuity equation using Reynold's Transport theorem in an elemental volume

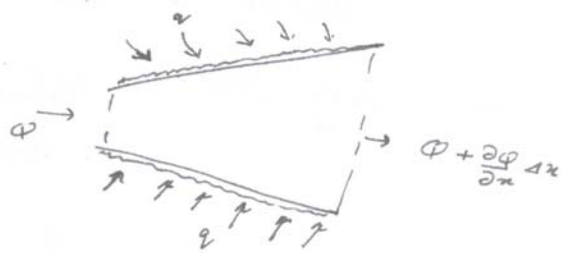
→ The conservation form of the continuity equation was :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - q = 0$$

→ The non-conservation form of the Saint Venant's continuity equation for cases having negligible lateral inflow :

$$V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = 0$$





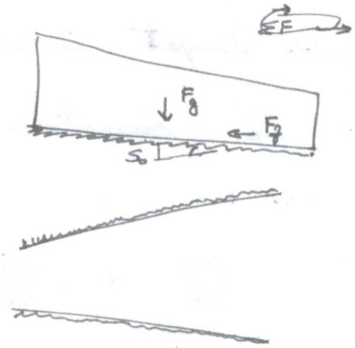
Saint Venant's Momentum equation

Again for the same control volume we use RTT

$$\sum \vec{F} = \frac{d}{dt}\iiint_{cv} \vec{v}\,\rho\, dV + \iint_{cs} \vec{v}\,\rho\,\vec{v}.\hat{n}\, dA$$

→ For an unsteady non-uniform flow in a channel, the forces acting on the elemental control volume are:

* Gravity force F_g
* Frictional force F_f
* Contraction / Expansion force, F_e
* Wind shear force, F_w
* Net pressure force, F_p



$$\therefore \sum \vec{F} = F_g + F_f + F_e + F_w + F_p \longrightarrow (1)$$

→ Now recall that we are dealing with one-dimensional fluid flow

→ The direction of flow will be same as the direction of net force.

The Gravity Force: Volume of fluid $= A\,\Delta x$

$\therefore$ Weight $= \rho g A \Delta x$

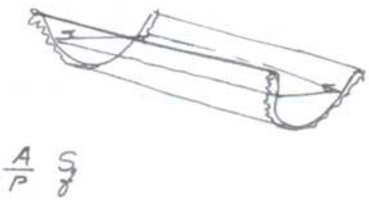
$\therefore$ The component of gravity force in the direction of flow:

$$F_g = \rho g A \Delta x\, S_0 \longrightarrow (2)$$

(Recall bed slope is small, $\therefore \sin\theta \approx S_0$)

Frictional force, F_f : The frictional force acts on the sides and bottom of the control volume.

If $\tau_0 \rightarrow$ bed shear stress
$P \rightarrow$ Wetted perimeter



This bed shear stress $\tau_0 = \rho g \frac{A}{P} S_f$

$S_f \rightarrow$ Friction slope from Manning's equation:

$$F_f = -\tau_0 P \Delta x$$

$$= -\rho g \frac{A}{P} P \Delta x S_f$$

$$\therefore F_f = -\rho g A S_f \Delta x \longrightarrow (3)$$

(Recall from open channel hydraulics, you may already have studied bed shear stress $\tau_0 = \rho g R S_f$
$R \rightarrow$ hydraulic radius).

Force due to contraction / expansion, F_e : These forces are due to sudden contraction or expansion of the channel. Mainly they are eddy losses that are function of velocity head.

Let us suggest eddy loss slope, $S_e = \frac{K_e}{2g} \frac{\partial}{\partial x}\left(\frac{Q}{A}\right)^2$

where $K_e \rightarrow$ non-dimensional expansion or contraction coefft.

$K_e \rightarrow$ −ive for channel expansion
$K_e \rightarrow$ +ive for channel contraction

The drag forces or eddy forces, $F_e = -\rho g A S_e \Delta x$

Wind shear force, F_w : The frictional resistance of wind against the free surface of water.

$$F_w = \tau_w \, B \, \Delta x$$

This wind shear force $F_w = - W_f \, B \, \rho \, \Delta x$

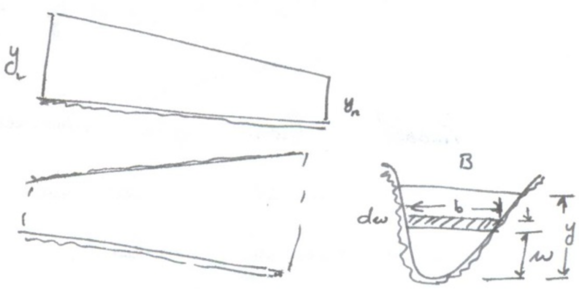
where $W_f = C_f \, |V_r| \, \frac{V_r}{2}$ → The wind shear factor

C_f → shear stress coefficient

V_r → velocity of fluid relative to the air boundary.

Pressure force, F_p :

We have to find the unbalanced pressure force in the control volume.



On the left boundary.
the hydrostatic pressure force = Hydrostatic Pressure × Area

Consider the elemental area = $b\,dw$

Hydrostatic force on this area = $\rho g (y - w) \, b \, dw$

∴ Hydrostatic pressure force on the entire left cross sectional area

$$F_{pl} = \int_0^y \rho g (y - w) \, b \, dw$$

III^{ly} on the right side we can now suggest

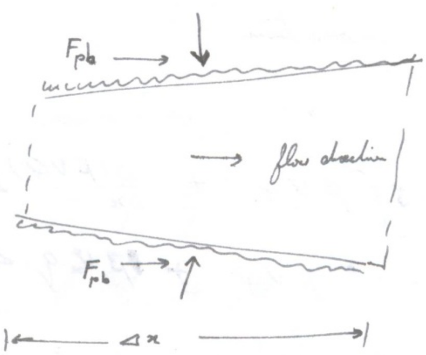
$$F_{pr} = \left(F_{pl} + \frac{\partial F_{pl}}{\partial x} \Delta x \right)$$

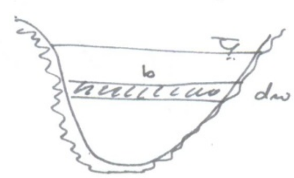
$$\therefore \frac{\partial F_{pl}}{\partial x} = \int_0^y \rho g \frac{\partial y}{\partial x} \, b \, dw + \int_0^y \rho g (y - w) \frac{\partial b}{\partial x} \, dw$$

i.e. $$\frac{\partial F_{pl}}{\partial x} = \rho g A \frac{\partial y}{\partial x} + \int_0^y \rho g (y-w) \frac{\partial b}{\partial x}\, dw$$

→ The pressure force from the banks need to be accounted due to change in width of the channel

F_{pb} = force due to banks (here we consider the quantity in the direction of flow or opposite to it. The perpendicular quantities are not accounted due to one dimensional flow).





The channel changes width from left to right.

The rate of change of width $= \frac{\partial b}{\partial x}$

$$\therefore\ F_{pb} = \left[\int_0^y \rho g (y-w) \frac{\partial b}{\partial x}\, dw\right] \Delta x$$

$$\therefore\ F_p = F_{pl} - \left(F_{pl} + \frac{\partial F_{pl}}{\partial x}\Delta x\right) + F_{pb}$$

$$= \int_0^y \rho g (y-w)\, b\, dw - \int_0^y \rho g (y-w)\, b\, dw$$

$$- \int_0^y \rho g \frac{\partial y}{\partial x}\, b\, dw\, \Delta x - \int_0^y \rho g (y-w) \frac{\partial b}{\partial x}\, dw\, \Delta x$$

$$+ \int_0^y \rho g (y-w) \frac{\partial b}{\partial x}\, dw\, \Delta x$$

i.e.
$$F_p = -\rho g A \frac{\partial y}{\partial x} \Delta x$$

$\therefore$ Now
$$\sum \vec{F} = \rho g A S_0 \Delta x - \rho g A S_f \Delta x - \rho g A S_e \Delta x - W_f B \rho \Delta x - \rho g A \frac{\partial y}{\partial x} \Delta x$$

In RTT:
$$\sum \vec{F} = \frac{d}{dt} \iiint_{cv} \vec{V} \rho \, d\forall + \iint_{cs} \vec{V} \rho \, \vec{V} \cdot \hat{n} \, dA$$

The net outflux of momentum
$$\iint_{cs} \vec{V} \rho \vec{V} \cdot \hat{n} \, dA \rightarrow$$
$$\rho \left[\beta V Q + \frac{\partial}{\partial x}(\beta V Q) \Delta x \right] - \rho \left(\beta V Q + \beta v_x q \Delta x \right)$$

$\beta \rightarrow$ Momentum correction factor. $\beta = \frac{1}{V^2 A} \iint v^2 dA$
(Recall from OCH).

$\rho \beta V Q \rightarrow$ Momentum flux entering the u/s end
$\rho \beta v_x q \rightarrow$ Momentum flux entering through lateral sides
$\rho \left(\beta V Q + \frac{\partial}{\partial x}(\beta V Q) \Delta x \right) \rightarrow$ Momentum leaving from right.

$$\therefore \iint_{cs} \vec{V} \rho \vec{V} \cdot \hat{n} \, dA = \rho \left[\frac{\partial}{\partial x} (\beta V Q) - \beta v_x q \right] \Delta x$$

$v_x \rightarrow$ Component of lateral inflow velocity in x-direction

$\rightarrow$ The change in momentum stored inside the CV:
$$\frac{d}{dt} \iiint_{cv} \vec{V} \rho \, dV = \frac{\partial}{\partial t}(\rho A \Delta x \, V) = \rho \frac{\partial Q}{\partial t} \Delta x$$

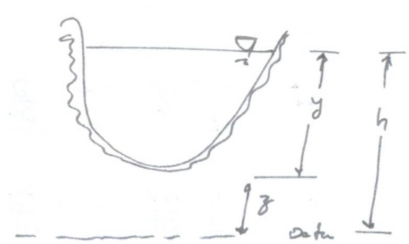
The momentum equation becomes:

$$\rho g A S_0 \Delta x - \rho g A S_f \Delta x - \rho g A S_e \Delta x - W_f B \rho \Delta x - \rho g A \frac{\partial y}{\partial x} \Delta x = \rho \frac{\partial Q}{\partial t} \Delta x + \rho \left[\frac{\partial}{\partial x}(\beta V Q) - \beta v_x q \right] \Delta x$$

We know $V = \frac{Q}{A}$.

Water surface elevation

$$h = y + z$$



→ The conservation form of momentum equation is:

$$g A S_0 - g A S_f - g A S_e - W_f B - g A \frac{\partial y}{\partial x} = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta Q^2}{A}\right) - \beta v_x q$$

or

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + g A \left(\frac{\partial y}{\partial x} - S_0 + S_f + S_e \right) - \beta v_x q + W_f B = 0$$

$$\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x}$$

Now $\frac{\partial z}{\partial x} = -S_0$, $\quad \therefore \frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} - S_0$

or

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + g A \left(\frac{\partial h}{\partial x} + S_f + S_e \right) - \beta q v_x + W_f B = 0$$

By neglecting Eddy losses wind shear, and lateral inflow, we can develop non-conservation form of momentum equation for a unit cross sectional width

$$A = y, \qquad Q = Vy$$